Halfplane Intersection:
Recall, each line in plane defines two halfspaces:
\[ l: y = ax + b \]
\[ h^+: y \geq ax + b \]
\[ h^-: y \leq ax + b \]
A halfspace is an (unbounded) convex set.

Given a set of halfspaces: \( H = \{ h_1, \ldots, h_n \} \),
their intersection \( \bigcap_i h_i \) is a (possibly unbounded / possibly empty) convex polygon.

Bounded

Unbounded

Empty
Representing lines (and more):

\( \mathbb{R}^2 \) (Line) \hspace{1cm} \mathbb{R}^d \) (Hyperplane)

**Explicit:**
\[
y = ax + b
\]

**Implicit:**
\[
f(x, y) = ax + by + c
\]

**Parametric:**
\[
(x(t), y(t))
\]

\[
(\mathbf{r} \cdot \sin(t), \mathbf{r} \cdot \cos(t))
\]

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**Halfplane Intersection:**

Given halfplanes \( H = \{h_1, \ldots, h_n\} \) construct \( H = \bigcup_i h_i \)

**Output:** Sequence of edges

\[
\langle 5, 1, 4, 2, 7, 6, 3 \rangle
\]

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**Divide and Conquer Algorithm:** \( \mathcal{O}(n \log n) \)

\[\text{Intersect}(H) \] 

- if \( |H| = 1 \) return \( h_1 \) [single halfspace]
- else partition \( H \) \( \bigcup \)

\[
|H_i| \leq \frac{n}{2}
\]

\[ I_1 \leftarrow \text{Intersect}(H_1); \quad I_2 \leftarrow \text{Intersect}(H_2) \]

**return merge \( (I_1, I_2) \)** How?
How to merge? Plane sweep

- At most 4 segments hit sweep line
- \( \leq n_1 + n_2 \) end pt events
  \( n_i = |H_i| \)
- \( \leq 2(n_1 + n_2) \) intersection events
- Boundaries are already sorted

\( O(n_1 + n_2) \) time

Overall Running Time:

\[ T(n) = 2T\left(\frac{n}{2}\right) + n \]

2 recursive calls on \( \frac{n}{2} \) halfspaces

\[ = O(n \log n) \quad \text{[see, e.g., CLRS]} \]

Special Case: Lower/Upper Envelopes

Upper Envelope

Lower Envelope
Envelopes of lines ~ Hull of points
Related?

Point-Line Duality
Lines in \( \mathbb{R}^2 \) are a lot like points:

- **2 degrees of freedom**: \( y = ax + b \)

- **degeneracy**: \( p : (a, b) \)

- **incidence**: \( p = l_1 \cap l_2 \)
  Two lines meet at a point

- \( l = p_1 \cup p_2 \)
  Two points join to form a line

**Dual Operator:**

Given:
- point \( p = (a, b) \), \( a, b \in \mathbb{R} \)
- line \( l : y = cx - d \), \( c, d \in \mathbb{R} \)

Dual:
- \( p^* \) is the line \( y = ax - b \)
- \( l^* \) is the point \( (c, d) \)
**Observations:**

- **Self-inverse:** $p^{**} = p$  
  $l^{**} = l$
- **Incidence:** $p$ lies on $l$  iff  $l^*$ lies on $p^*$

**Proof:**

\[ b = c \cdot a - d \quad \Leftrightarrow \quad d = a \cdot c - b \]

\[ l : y = cx - d \]

\[ l^* = (c, d) \]

\[ p^* : y = ax - b \]

**Order reversing:** $p$ lies above/ below $l$  iff  $p^*$ passes below/ above $l^*$

**Proof:**

\[ b > c \cdot a - d \quad \Leftrightarrow \quad d > a \cdot c - b \]

**Degeneracy:**

$p_1, p_2, p_3$ are collinear  iff  $p_1^*, p_2^*, p_3^*$ are coincident
Hulls and Envelopes:

Lemma:
Given a set $P = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^2$, the CCW order of points on $P$'s upper/lower hull is same as left-right order of segments in $P^*$'s lower/upper envelope.

Proof: (Sketch)
Consider edge $p_i p_j$ on upper hull of $\text{conv}(P)$.

Let $l$ be line $p_i p_j$ - All pts of $P$ lie on or below $l$.

$\iff$ (order reversal) - All lines of $P^*$ pass on or above point $l^*$.

$\iff$ $l^*$ is vertex of lower envelope.