CMSC 754 - Computational Geometry
Lecture 6: Halfplane Intersection + Duality

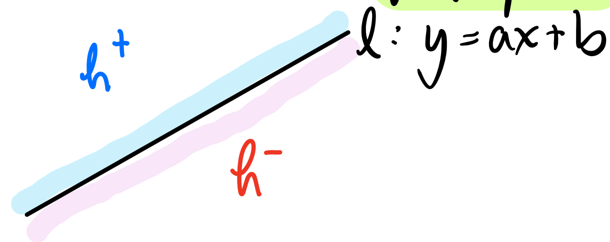
Halfplane Intersection:

Recall, each line in plane defines two halfspaces

$\ell: y = ax + b$

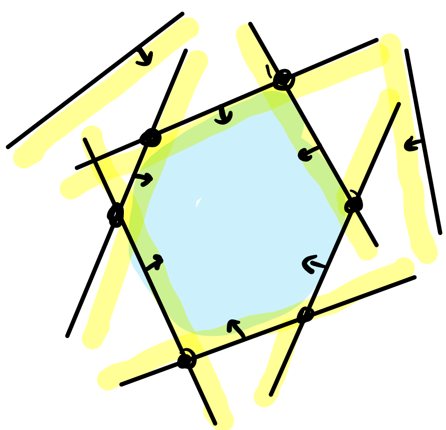
$h^+: y \geq ax + b$

$h^-: y \leq ax + b$

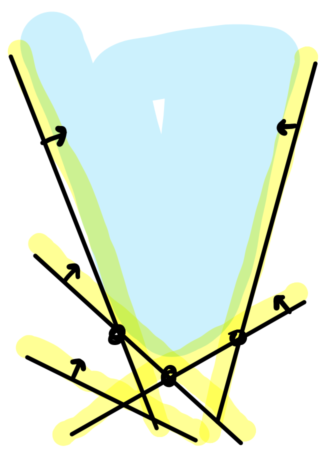


A halfspace is an (unbounded) convex set

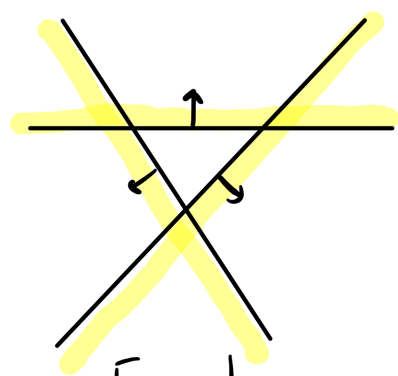
Given a set of halfspaces: $H = \{h_1, \dots, h_n\}$
their intersection $\bigcap_{i=1}^{n} h_i$ is a (possibly unbounded / possibly empty) convex polygon



Bounded



Unbounded

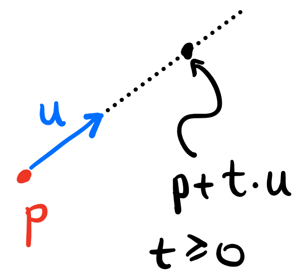


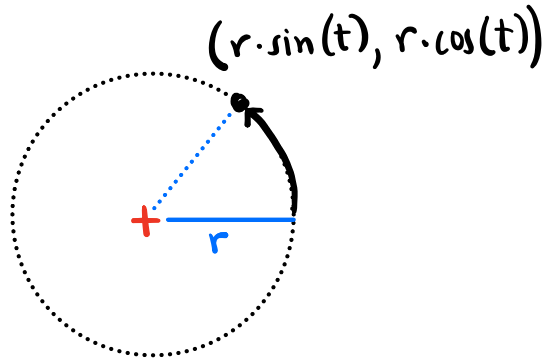
Empty

Representing lines (and more):

	$\mathbb{R}^2$ (Line)	$\mathbb{R}^d$ (Hyperplane)
Explicit: $y = f(x)$	$y = ax + b$	$x_d = \sum_{i=1}^{d-1} a_i x_i + b$
Implicit: $f(x,y) = 0$	$f(x,y) = ax + by + c$	$f(x_1, \ldots, x_d) = \sum_{i=1}^{d} a_i x_i + b$

Parametric: $(x(t), y(t))$



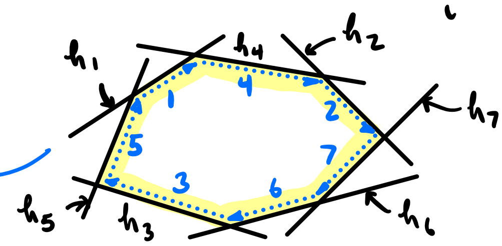


Halfplane Intersection:

Given halfplanes $H = \{h_1, \ldots, h_n\}$ construct $\mathcal{H} = \bigcap_i h_i$

Output: Sequence of edges

$\langle 5, 1, 4, 2, 7, 6, 3 \rangle$



Divide and Conquer Algorithm: $O(n \log n)$

Intersect (H) {

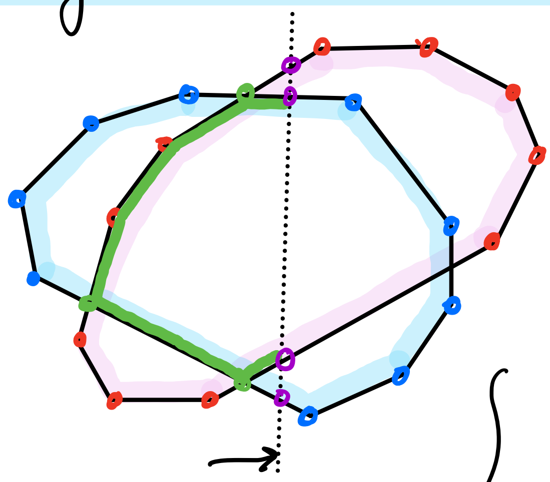
- if(| H | = 1) return h_1 [single halfspace]
- else

  partition H → H_1, H_2 $|H_i| \leq n/2$

  $I_1 \leftarrow$ Intersect (H_1); $I_2 \leftarrow$ Intersect (H_2)

  return merge (I_1, I_2) ← How?

How to merge? Plane sweep



$O(n_1 + n_2)$ time

- At most 4 segments hit sweep line
- $\leq n_1 + n_2$ end pt events
  $n_i = |H_i|$
- $\leq 2(n_1+n_2)$ intersection events
- Boundaries are already sorted

Overall Running Time:

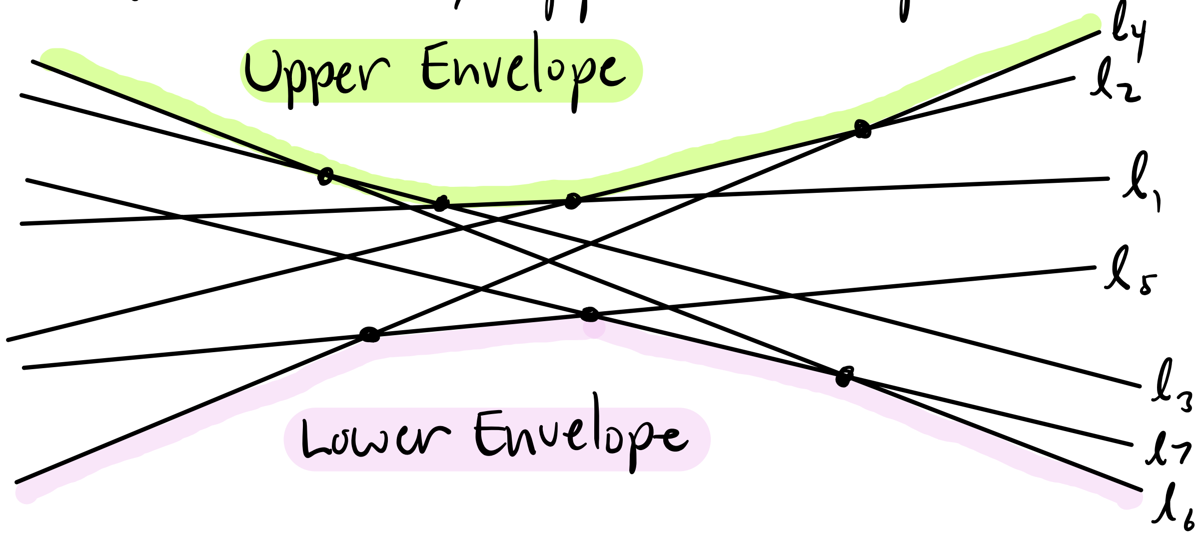
$$T(n) = 2T(n/2) + n$$

2 recursive calls on n/2 halfspaces

merge in linear time

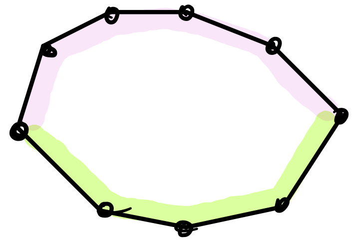
$$= O(n \log n)$$ [see, eg., CLRS]

Special Case: Lower/Upper Envelopes



Envelopes of lines ~ Hull of points

Related?

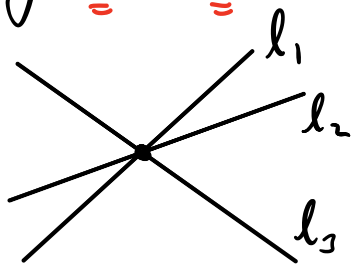


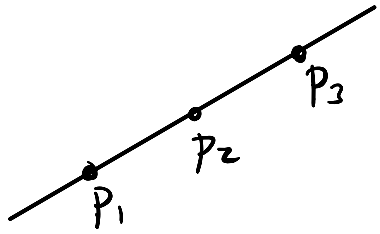
Point-Line Duality

Lines in $\mathbb{R}^2$ are a lot like points:

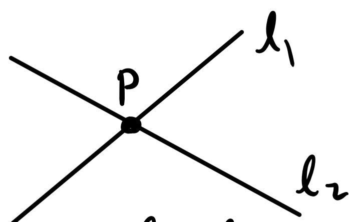
2 degrees of freedom: $y = a x + b$ $\qquad$ $p: (a, b)$

degeneracy:



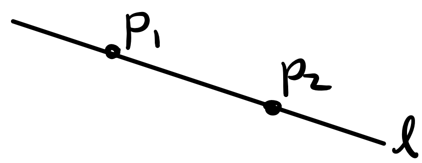


incidence:



$p = l_1 \wedge l_2$

Two lines meet at a point



$l = p_1 \vee p_2$

Two points join to form a line

Dual Operator:

Given point $p = (a, b)$ $\qquad a, b \in \mathbb{R}$

line $l: y = cx - d$ $\qquad c, d \in \mathbb{R}$

Dual p^* is the line $y = a \cdot x - b$

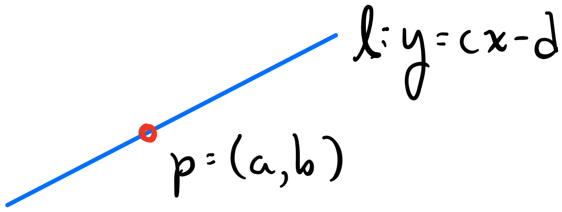
l^* is the point (c, d)

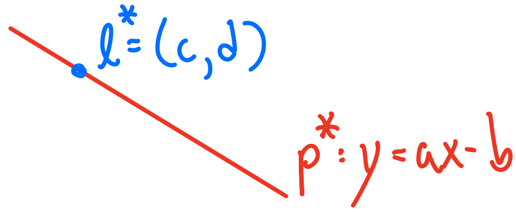
Observations:

Self-inverse: $p^{**} = p \quad \ell^{**} = \ell$

Incidence: p lies on ℓ iff ℓ^* lies on p^*

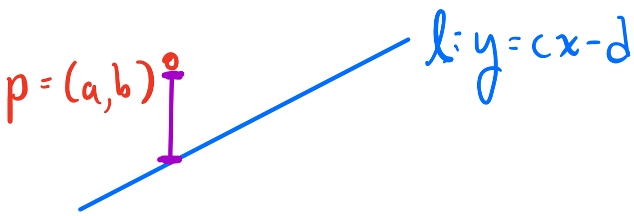
Proof: $b = c \cdot a - d \iff d = a \cdot c - b$

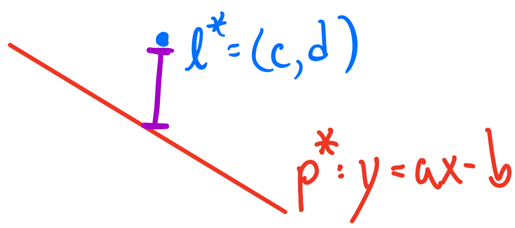




Order reversing: p lies above/below ℓ iff p^* passes below/above ℓ^*

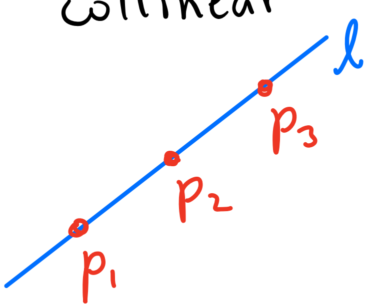
Proof: $b > c \cdot a - d \iff d > a \cdot c - b$

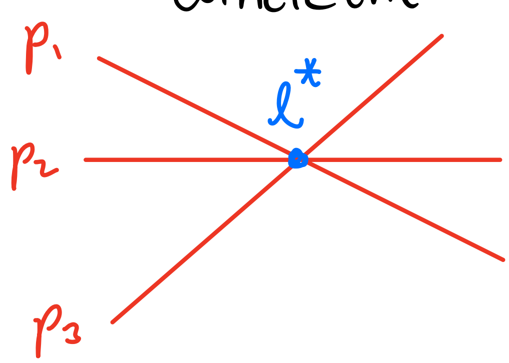




Degeneracy:

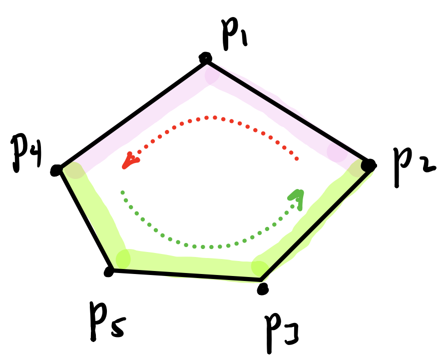
p_1, p_2, p_3 are collinear iff p_1^*, p_2^*, p_3^* are coincident

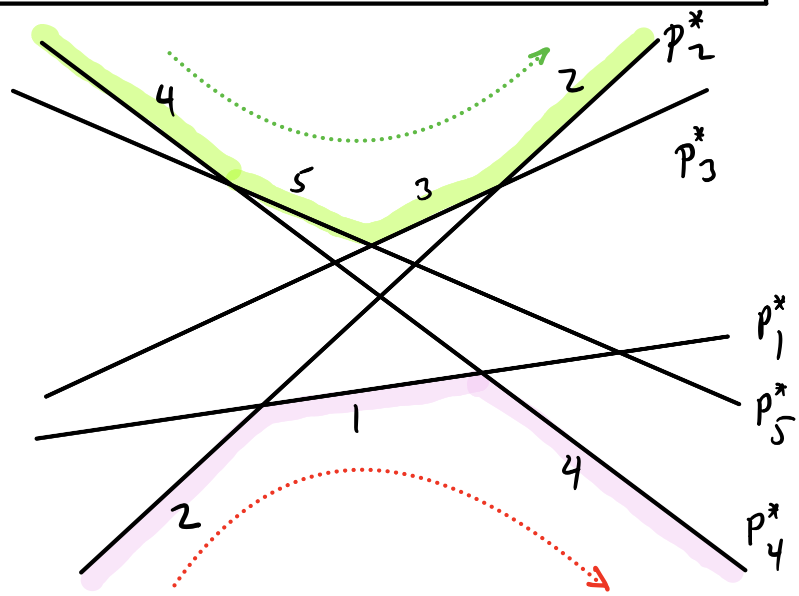




Hulls and Envelopes:

Lemma: Given a set $P = \{p_1, \dots, p_n\}$ in $\mathbb{R}^2$, the CCW order of points on P's upper/lower hull is same as left-right order of segments in P^*'s lower/upper envelope





Proof: (Sketch)

Consider edge $p_i p_j$ on upper hull of conv(P)

Let ℓ be line $\overleftrightarrow{p_i p_j}$ – All pts of P lie on or below ℓ

$\Leftrightarrow$ (order reversal) – All lines of P^* pass on or above point ℓ^*

$\Leftrightarrow$ ℓ^* is vertex of lower envelope