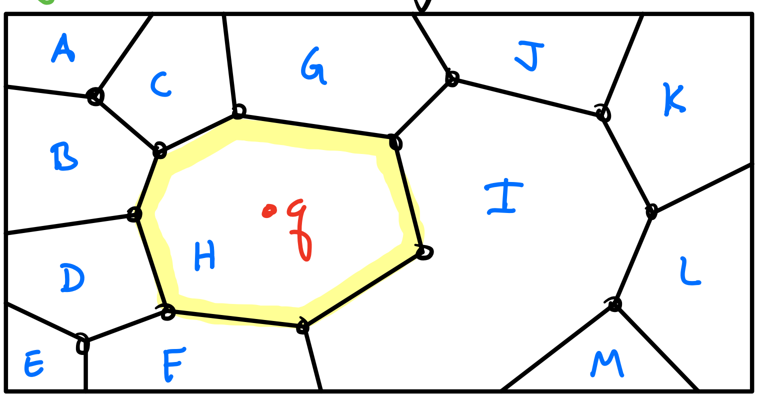
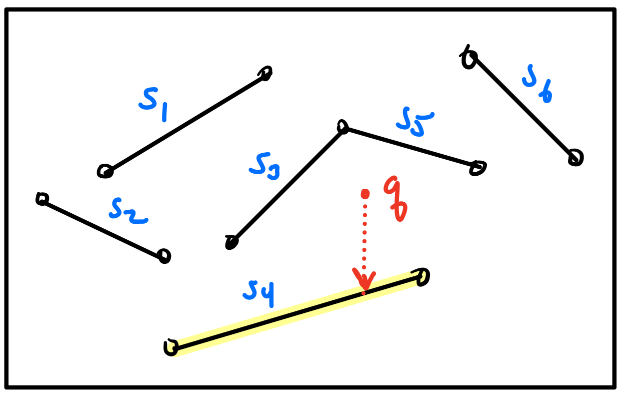
CMSC 754 - Computational Geometry
Lecture 9: Planar Point Location (via Trap. Maps)

Planar Point Location:

Given a subdivision of the plane (cell complex), build a data structure so that for any query pt, can find the cell containing it.

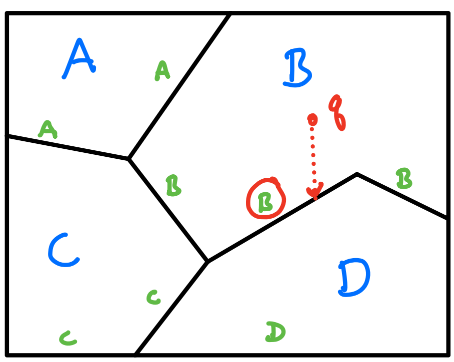




Vertical Ray Shooting:

Given a set of disjoint line segments in the plane, build a data structure s.t. given any query pt q, can report the segment immediately below.

Ray Shooting ⇒ Point Location



Label each segment with region just above

Data structure for vertical ray shooting:

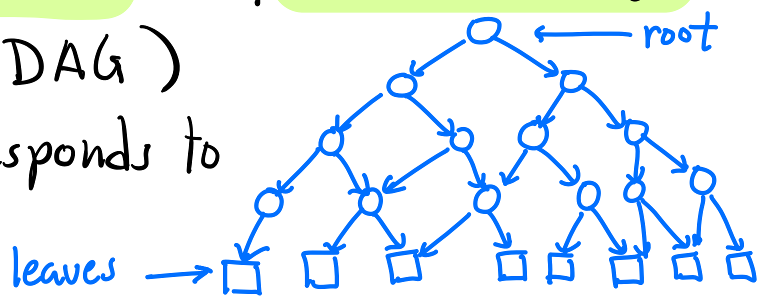
Approach: Build **trapezoidal map** + **ray shooting** structure simultaneously

$S = \{s_1, \dots, s_n\}$ Randomly permuted $\rightarrow \mathcal{T}(S)$
$S_i = \{s_1, \dots, s_i\}$ $\rightarrow$ Partial map $\mathcal{T}(S_i) = \mathcal{T}_i$

Recall: In expectation, each insertion results in **$O(1)$ changes** to structure.

Overview:

- Rooted **binary tree** with **shared subtrees** (a rooted DAG)
- Each **leaf** corresponds to a **trapezoid**

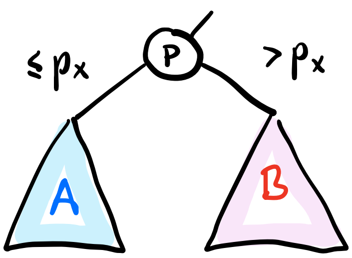


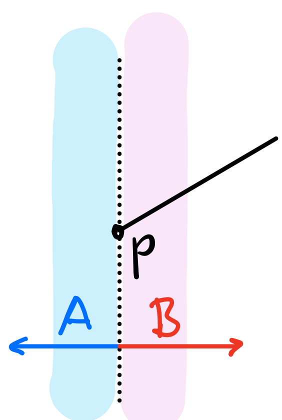
- Each trapezoid occurs **exactly once** as leaf

- **Internal nodes** – two types

x-Node:

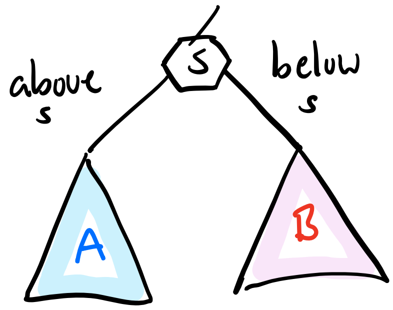
labeled with an endpt p

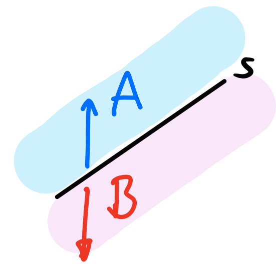




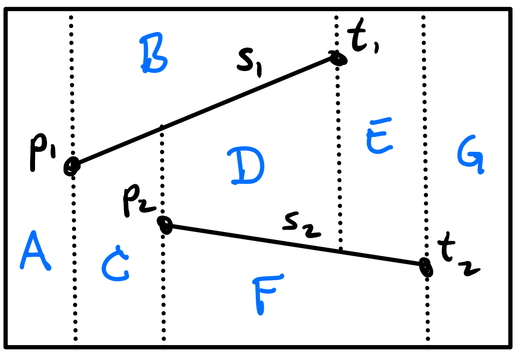
y-Node:

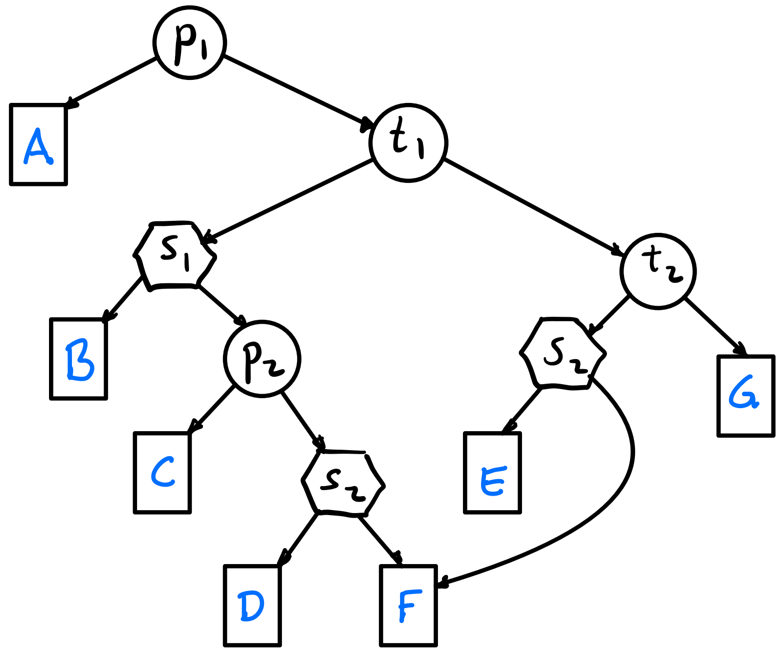
Labeled with a segment s



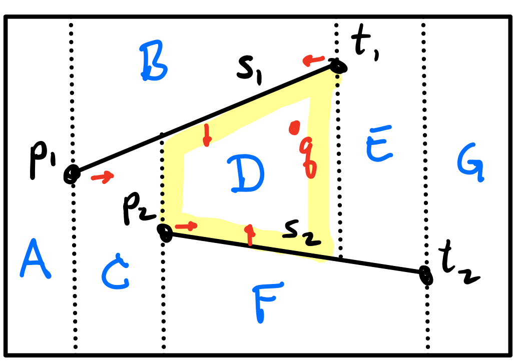


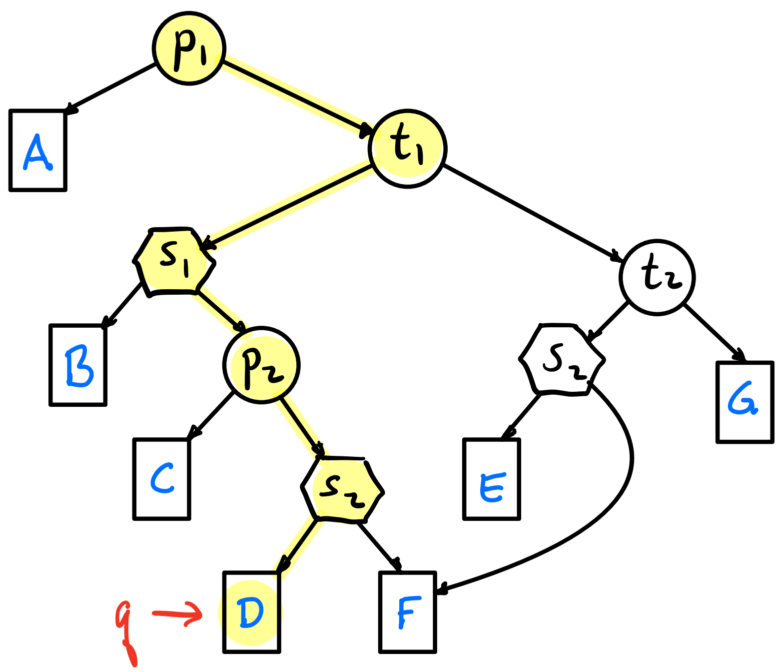
Example:





Query processing:



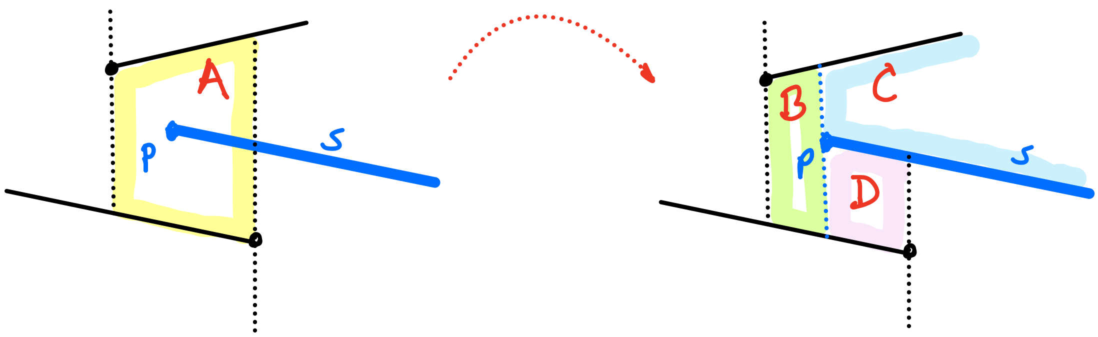


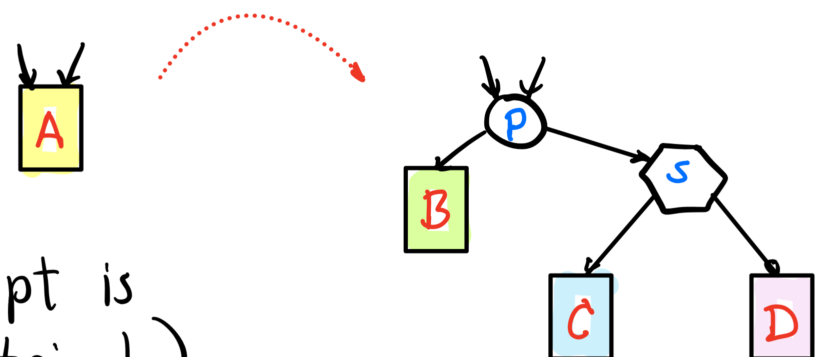
Incremental Construction:

- As segments are added: $s_1, s_2, \ldots, s_i$
  we build structure for $\mathcal{T}(S_1), \mathcal{T}(S_2) \ldots \mathcal{T}(S_i)$

- Update process:
  - Each added segment causes some trapezoids to go away + others created
  - We replace old leaves with new structures
  - By sharing, only one leaf per trapezoid

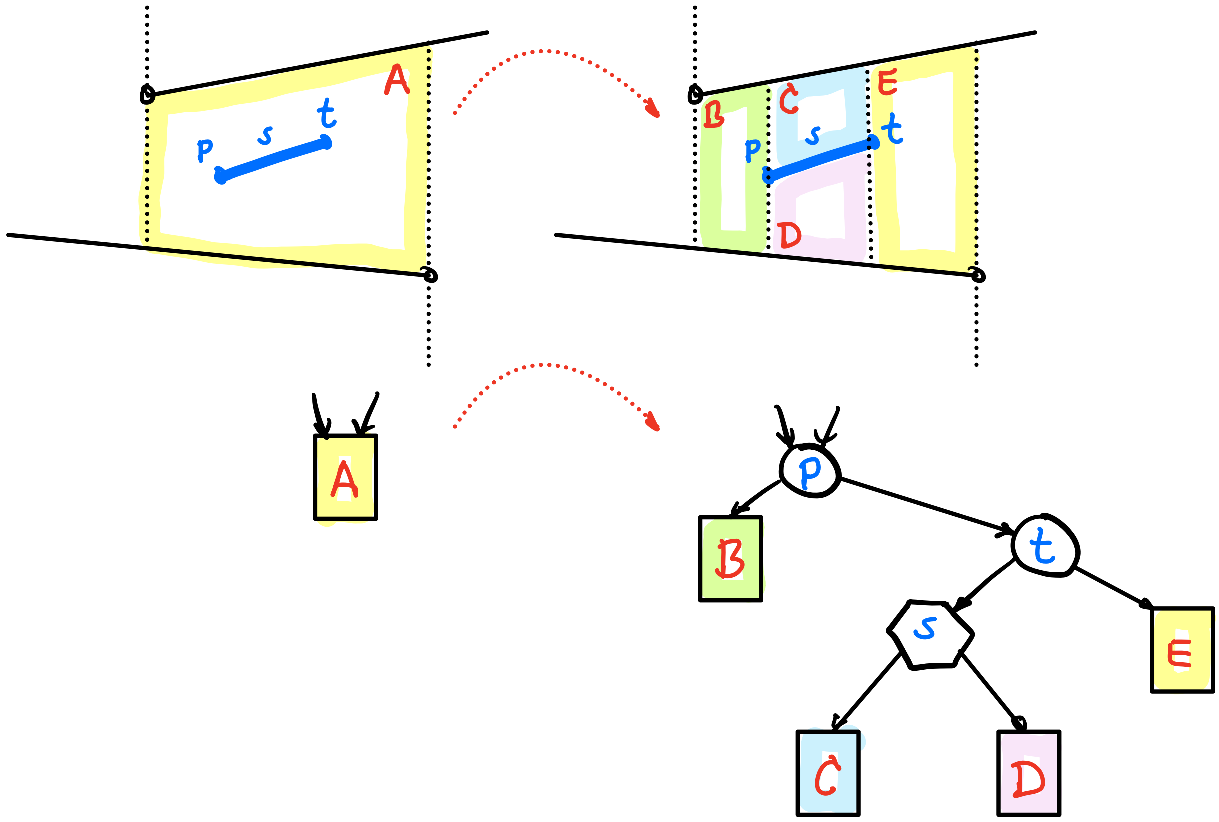
1: Single endpt in trapezoid (left or right):



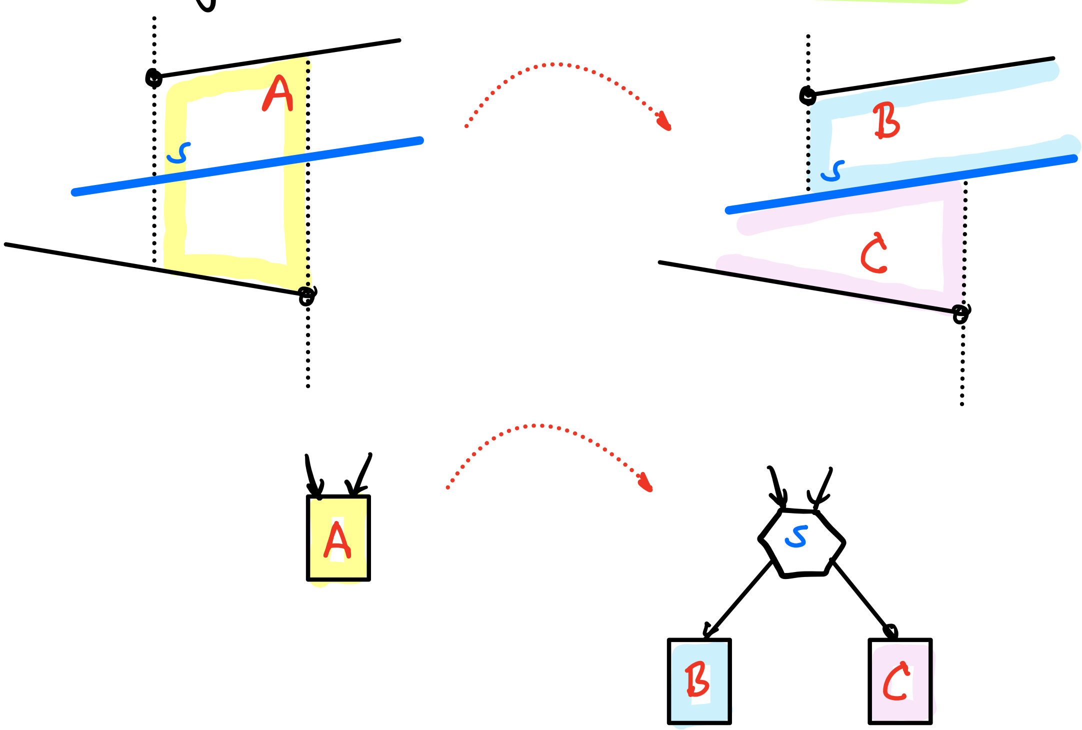


(Right endpt is symmetrical)

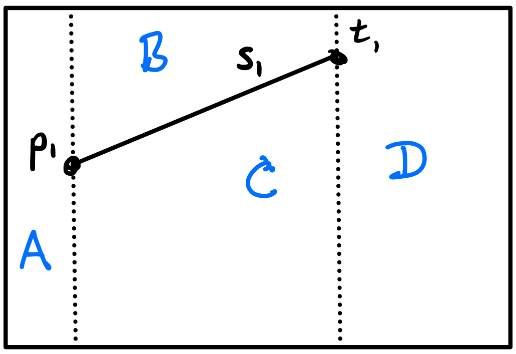
2: Two segment endpts in same trapezoid

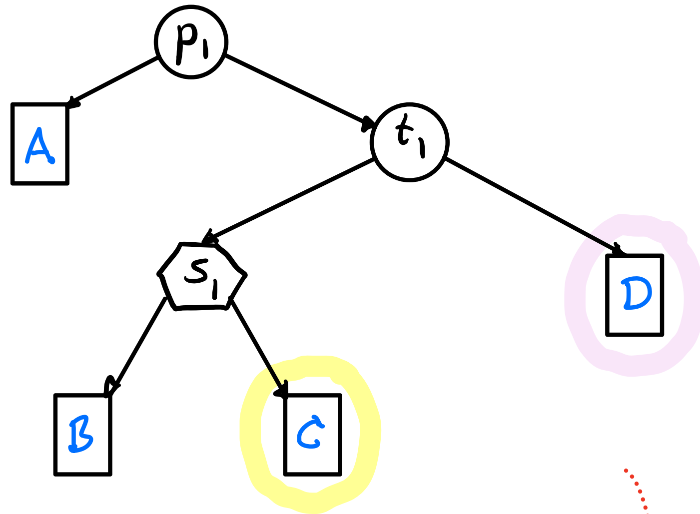


3: No segment endpt in trapezoid

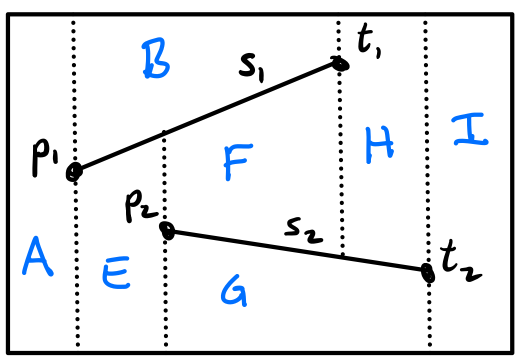


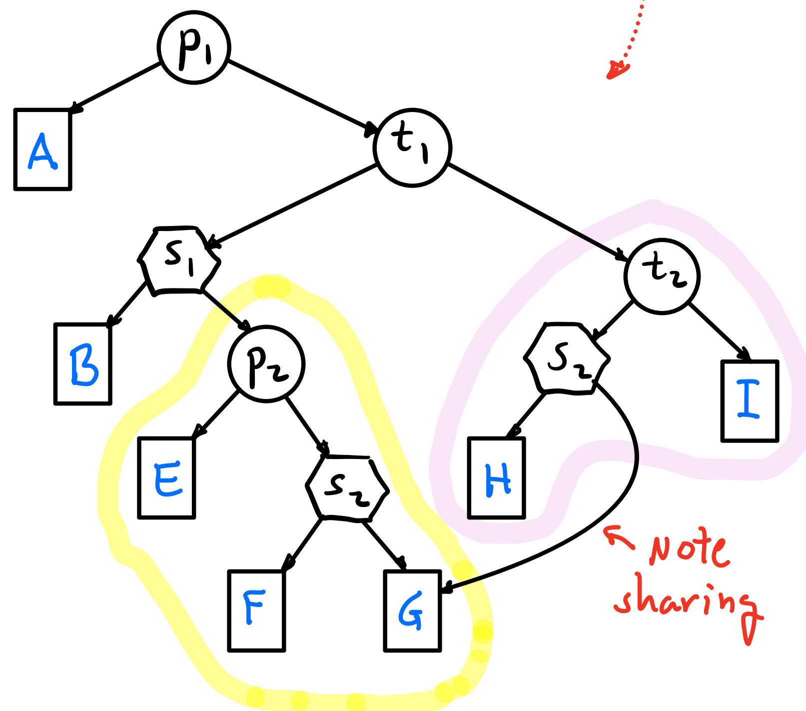
Example:





insert $s_2 = \overline{p_2 t_2}$





Analysis:

Will show if segs are inserted in random order, expected space is $O(n)$ + expected search time for any fixed query pt is $O(\log n)$

Thm: The expected case space is $O(n)$

Proof: Last lecture we showed that expected no. of changes is $O(1)$ per seg $\Rightarrow$ total changes $O(n)$

Number of new nodes $\sim$ number of changes
$\Rightarrow$ final expected size is $O(n)$

Thm: Given a fixed query pt $q \in \mathbb{R}^2$, the expected search depth for q is $O(\log n)$

[Huh? Does this imply that depth of search tree is $O(\log n)$ in expectation?
No - But see our text for a proof of this.]

Proof:

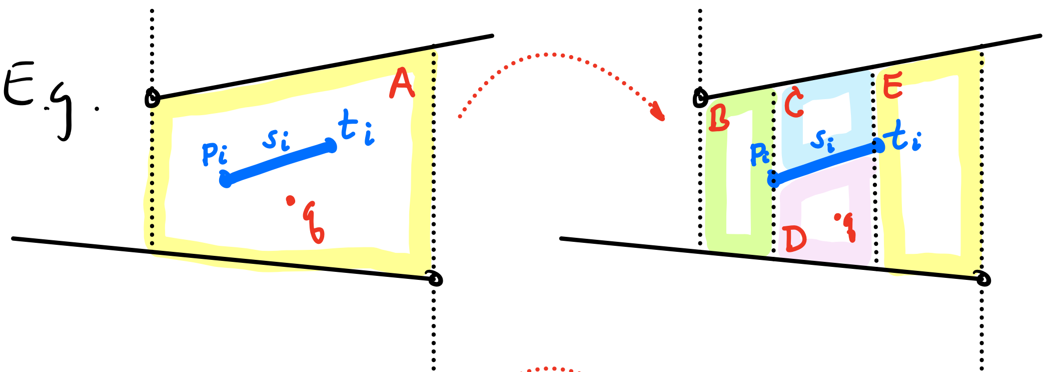
- Let q be any fixed query pt.
- Let $\Delta_i(q)$ be the trapezoid containing q after the insertion of s_i $(1 \leq i \leq n)$
  - Note: Sometimes $\Delta_i(q) = \Delta_{i-1}(q)$ (s_i had no impact)
    - What if $\Delta_i(q) \neq \Delta_{i-1}(q)$?

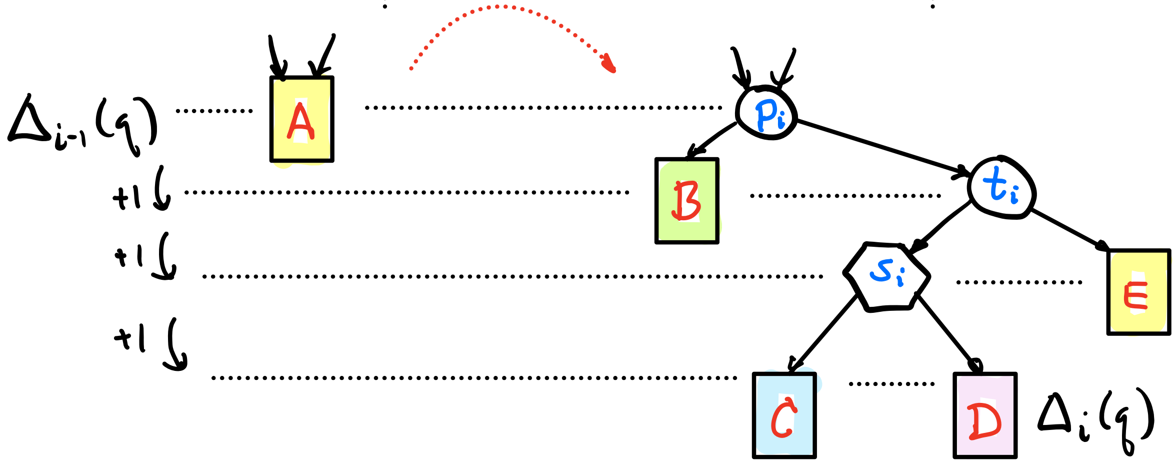
- For $1 \le i \le n$, let $X_i(q) = \begin{cases} 1 & \text{if } \Delta_i(q) \neq \Delta_{i-1}(q) \\ 0 & \text{o.w.} \end{cases}$

- If $X_i(q) = 1$, $\text{depth}(\Delta_i) \le 3 + \text{depth}(\Delta_{i-1})$

(Else depth doesn't change)

E.g.





Let $D(q)$ the expected depth of q's trapezoid in the final structure.

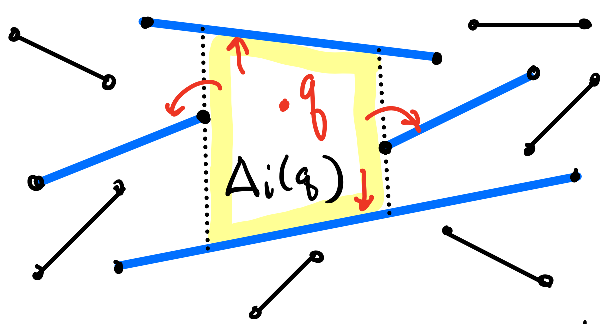
$$D(q) \le 3 \sum_{i=1}^{n} E(X_i(q))$$

$$= 3 \sum_{i=1}^{n} \text{Prob}(\Delta_i(q) \neq \Delta_{i-1}(q))$$

- We assert that $\text{Prob}(\Delta_i(q) \neq \Delta_{i-1}(q)) \leq 4/i$

  - Backwards analysis:
    - Each of the existing i segs is equally likely to be last (prob $= 1/i$)

    - $\Delta_i(q) \neq \Delta_{i-1}(q)$ iff last segment is one of the 4 segments incident to $\Delta_i(q)$



$\Rightarrow \text{Prob}(\Delta_i(q) \neq \Delta_{i-1}(q)) \leq 4/i$

- Substituting: Expected depth of q's trapezoid

$$D(q) \leq 3\sum_{i=1}^{n} E(X_i(q)) = 3\sum_{i=1}^{n} \text{Prob}(\Delta_i \neq \Delta_{i-1})$$

$$\leq 3 \cdot \sum_{i=1}^{n} 4/i = 12 \sum_{i=1}^{n} 1/i \quad \text{(Harmonic series)}$$

$$\cong 12 \ln n = O(\log n) \qquad \square$$

Summary:

- Last time we showed that randomized incremental alg. took $O(1)$ time in expectation per segment, ignoring time to locate left end pt.

- Today, we presented a data structure with query time $O(\log n)$ for pt location

$\Rightarrow$ Total expected construction time is:

$$T(n) = \sum_{i=1}^{n} \Big(\underbrace{(\log i)}_{\text{locate left end pt}} + \underbrace{1}_{\text{update structure}} \Big)$$

$= O(n \log n)$

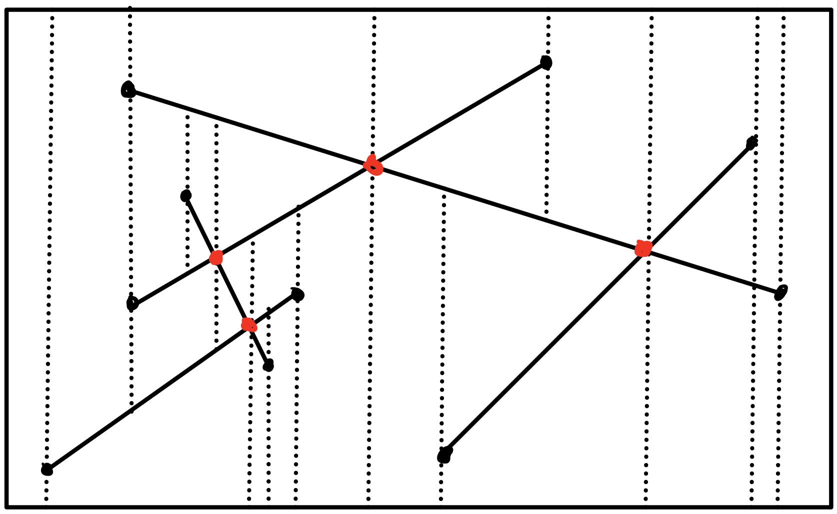
- Space + Query time are in expectation
  - Can we guarantee them?

  $\rightarrow$ Yes: Just rebuild if things go wrong (Increases expected construct time slightly, but still $O(n \log n)$.)

(See text for details)

Line segment intersection (Revisited):

- Can extend trap. maps to intersecting segs.



- Randomized construction can be easily generalized.

Expected time: $O(n \log n + m)$

where $m = \#$ of intersections

This beats plane sweep! $O((n+m)\log n)$