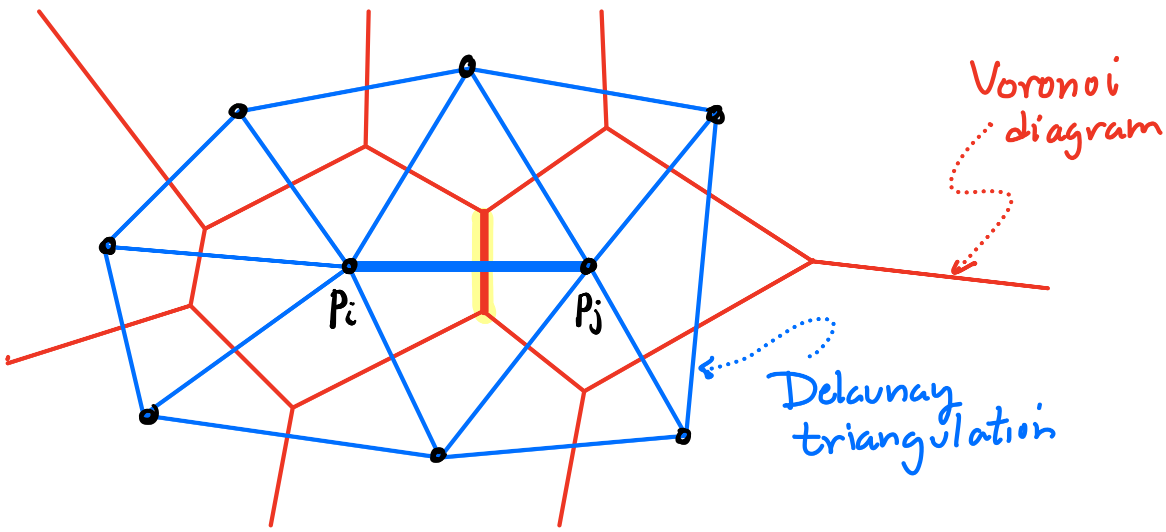
CMSC 754 - Computational Geometry
Lecture 11: Delaunay Triangulations (Properties)

Last lecture - Voronoi Diagrams

This - The dual structure - Delaunay Triangulations



Delaunay Triangulation:

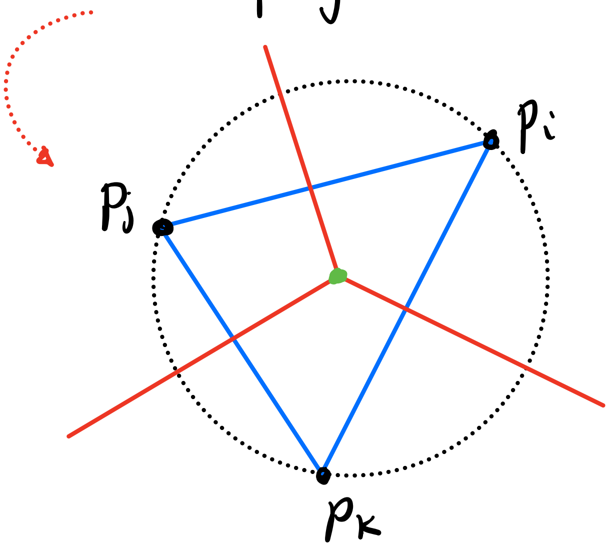
Given a set $P = \{p_1, \ldots, p_n\}$ of sites in $\mathbb{R}^2$, the Delaunay Triangulation is the cell complex whose vertices are sites & there is an edge $\overline{p_i p_j}$ iff $V(p_i)$ & $V(p_j)$ share a common edge. Called $DT(P)$

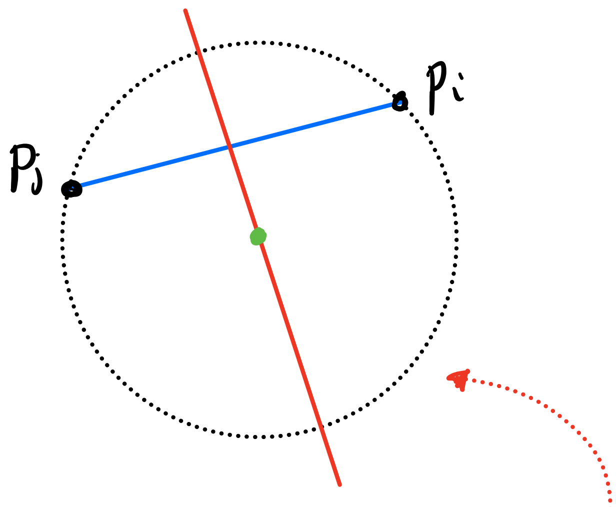
Properties:

Triangulation: If general position (no four sites cocircular), the internal faces are all triangles

Hull: The boundary of the external face is the boundary of conv(P)

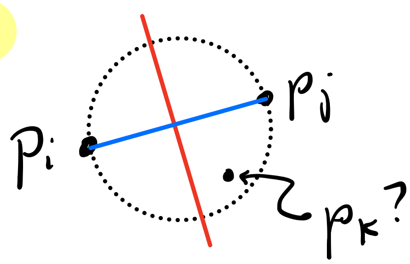
Circumcircle: The circumcircle of any triangle is empty (no sites in its interior)





Empty Circle: Sites p_i + p_j are adjacent in DT(P) iff there is an empty circle through p_i + p_j.

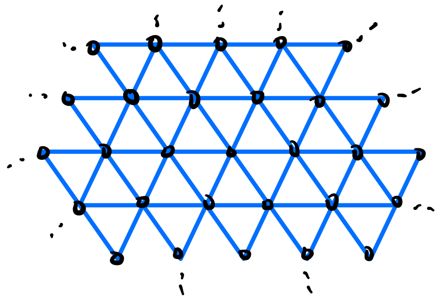
Closest Pair: The closest pair of sites are Delaunay neighbors



- Consider the circle with diameter $\overline{p_i p_j}$.
- No site p_k can lie within
- Apply empty circle prop.

Combinatorial Complexity:

By applying Euler's formula, there are at most $2n$ triangles and at most $3n$ edges



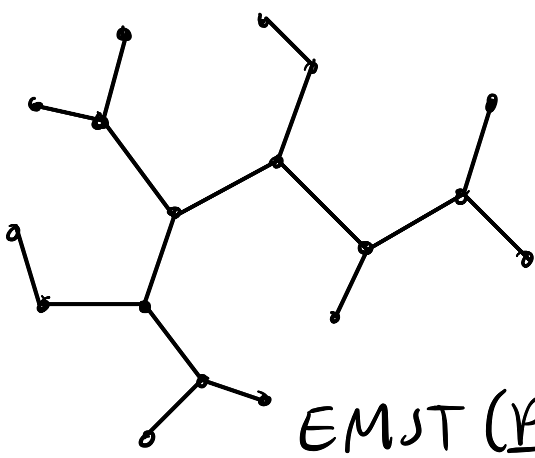
[In $\mathbb{R}^d$, size is $O(n^{\lceil d/2 \rceil})$]

Euclidean Minimum Spanning Tree: (EMST)

Euclidean graph: Complete graph on vertex set $P = \{p_1, \dots, p_n\}$, where edge weight is Euclidean distance $(w(p_i, p_j) = \|p_i - p_j\|)$

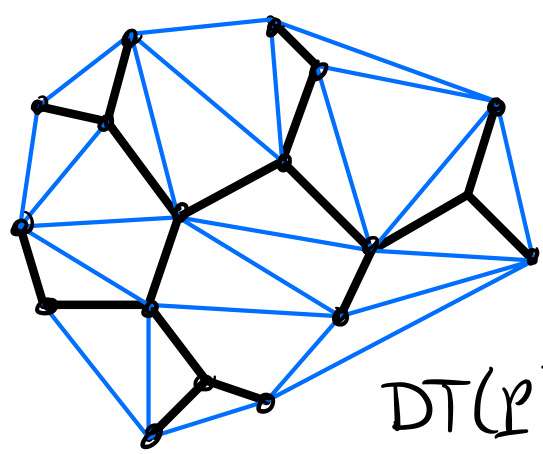
$EMST(P)$ = MST of Euclidean graph (lowest weight tree spanning P)

$$\boxed{\text{Thm: } EMST(P) \subseteq DT(P)}$$



EMST(P)

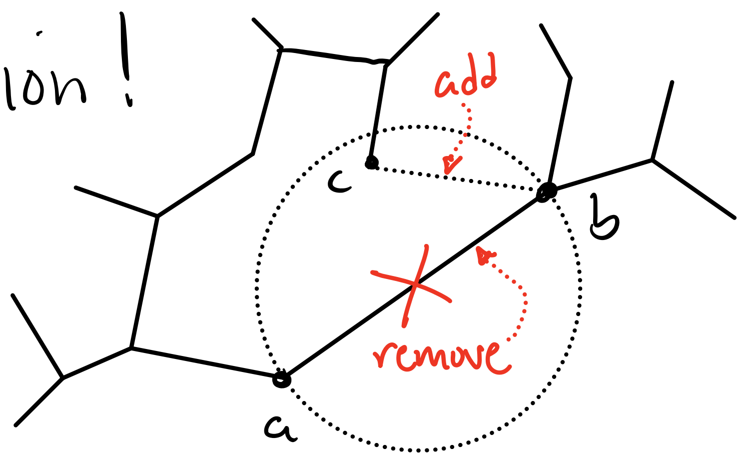
$\subseteq$



DT(P)

Proof: (Contradiction)

- Suppose some edge $\overline{ab} \in EMST(\mathcal{P})$ but not in $DT(\mathcal{P})$
- Empty circle $\Rightarrow$ circle with diameter $\overline{ab}$ contains site c
- $\|ac\| < \|ab\|$
  $\|bc\| < \|ab\|$
- Can remove $\overline{ab}$ from EMST + replace with either $\overline{ac}$ or $\overline{bc}$ to produce a spanning tree of lower weight
- Contradiction!



$\square$

Minimum Weight Triangulation: No!

$MWT(\mathcal{P})$ = triangulation of $\mathcal{P}$ whose sum of edge lengths is minimum

Generally $MWT(\mathcal{P}) \neq DT(\mathcal{P})$

Notation: Given graph $G=(V,E)$ and vertices $u,v \in V$, let $\delta_G(u,v)$ = shortest path distance in G from u to v.

Spanner Properties:

Given a graph G and $t \geq 1$, a t-spanner is a subgraph G' of G on same vertex set s.t. $\forall\, u,v \in V$,

$$\delta_{G'}(u,v) \leq t \cdot \delta_G(u,v)$$

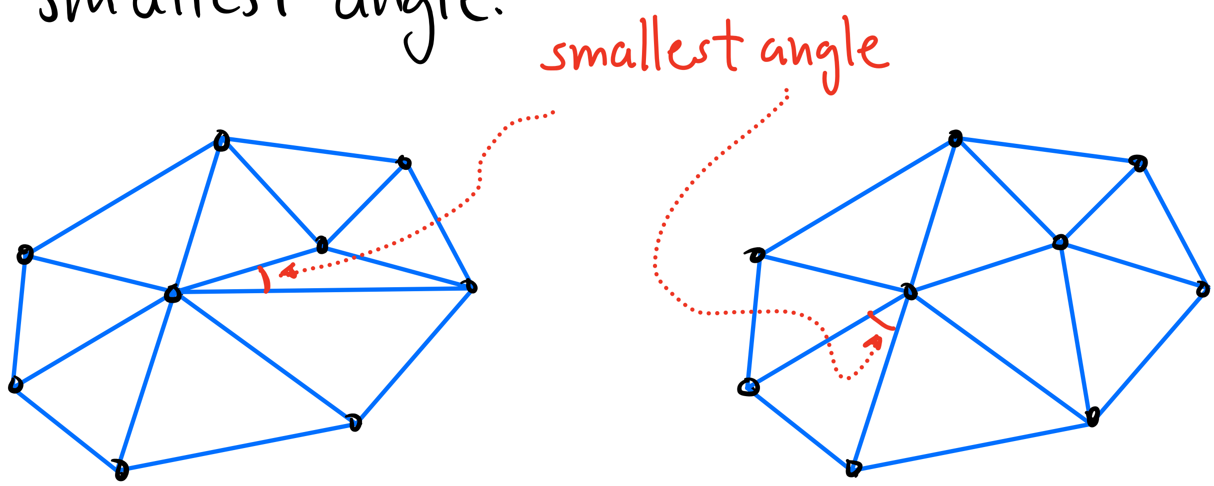
(Path lengths don't stretch too much)

Theorem (Keil + Gutwin, '92) Given a set P of sites in the plane, $DT(P)$ is a $4\pi\sqrt{3}/9 \approx 2.418$ spanner of the Euclidean graph. That is, $\forall\, p,q \in P$

$$\delta_{DT(P)}(p,q) \leq \frac{4\pi\sqrt{3}}{9} \cdot \|p-q\|$$

Avoids skinny triangles:

Let $\underline{P}$ be a set of sites in the plane. Among all possible triangulations of $\underline{P}$, DT($\underline{P}$) maximizes the size of the smallest angle.



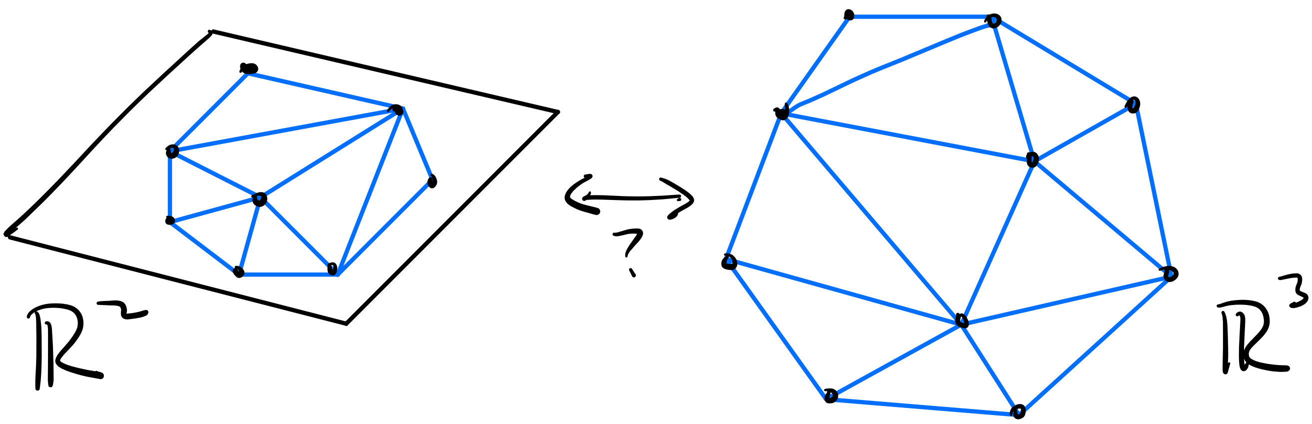
other triangulation

DT($\underline{P}$)

> **Thm:** If all angles of all triangles are ordered small to large, DT($\underline{P}$) is the largest lexicographically compared to all triangulations of $\underline{P}$.

(See full lecture notes)

Relationship to polytopes in $\mathbb{R}^{d+1}$



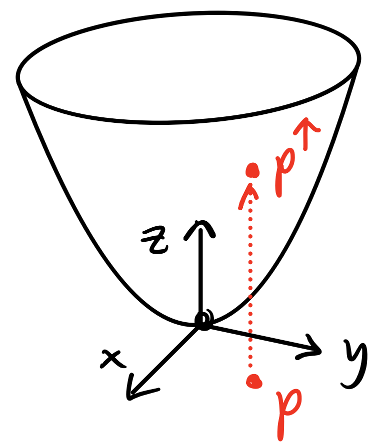
Delaunay triangulation in $\mathbb{R}^d$ is the projection of a lower convex hull in $\mathbb{R}^{d+1}$

Voronoi diagram in $\mathbb{R}^d$ is the projection of a lower envelope of hyperplanes in $\mathbb{R}^{d+1}$

→ We'll prove the first only: $\mathbb{R}^2 \to \mathbb{R}^3$

Consider the paraboloid:

$$z = f(x,y) = x^2 + y^2$$



Given $p = (p_x, p_y)$, define $p^\uparrow$ to be $(p_x, p_y, p_x^2 + p_y^2)$

Lemma:

Three pts $p,q,r \in \mathbb{R}^2$ have an empty circumcircle w.r.t. P $\Leftrightarrow$ Three pts $p^\uparrow, q^\uparrow, r^\uparrow$ lie on plane h with all pts of $P^\uparrow$ above

- Let $c = (c_x, c_y)$ be center of circumcircle through p, q, r & let r be its radius

- The plane tangent to paraboloid at $c^\uparrow$ is:

$$z = 2c_x \cdot x + 2c_y \cdot y - (c_x^2 + c_y^2)$$

- Shift this plane up by distance r^2:

$$h: \quad z = 2c_x \cdot x + 2c_y \cdot y - (c_x^2 + c_y^2) + r^2$$

- All 3 lifted pts lie on this plane:

p_x on circle: $(p_x - c_x)^2 + (p_y - c_y)^2 = r^2$

$$\Leftrightarrow (p_x^2 - 2p_x c_x + c_x^2) + (p_y^2 - 2p_y c_y + c_y^2) = r^2$$

$$\Leftrightarrow p_x^2 + p_y^2 = 2c_x \cdot p_x + 2c_y \cdot p_y - (c_x^2 + c_y^2) + r^2$$

$$\Leftrightarrow p_z^{\uparrow} = 2c_x \cdot p_x^{\uparrow} + 2c_y\, p_y^{\uparrow} - (c_x^2 + c_y^2) + r^2$$

$\Leftrightarrow$ $p^{\uparrow}$ lies on plane h