CMSC 754 - Computational Geometry
Lecture 16 - Well-Separated Pair Decompositions

Geometric Approximations:

- Useful when exact computation is too costly
- Geometric inputs are "measurements" and often are uncertain.

So approximate solutions are fine.

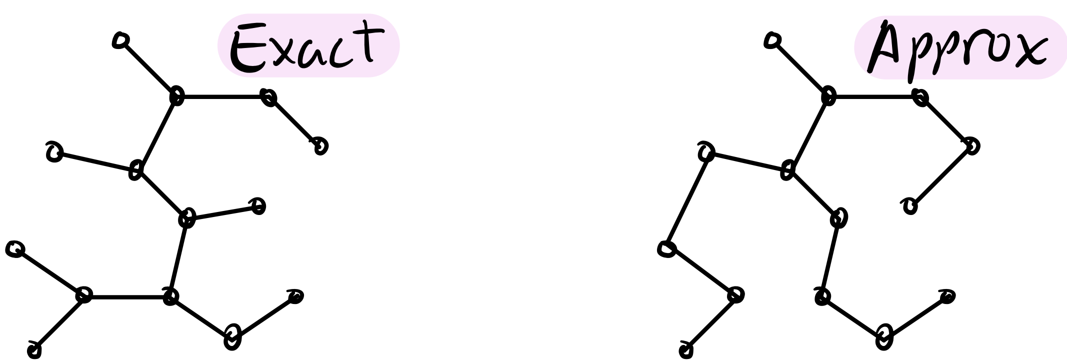
Examples:

Euclidean MST of pt set $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$

Exact: $O(n \log n)$ in $\mathbb{R}^2$

$O(n^{2-4/d})$ in $\mathbb{R}^d$ [Nearly quadratic]

Approx: Given $\varepsilon > 0$, compute a spanning tree of weight $\leq (1+\varepsilon) \cdot EMST(P)$



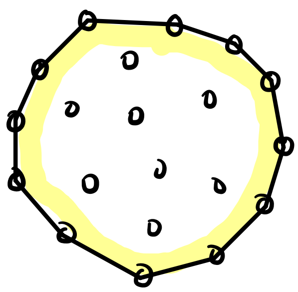
Convex Hull of a set $P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$

Exact: $O(n \log n)$ in $\mathbb{R}^2$

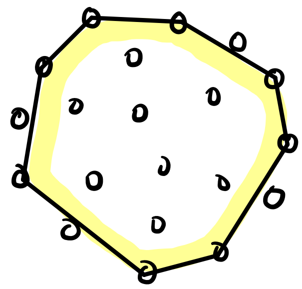
$O(n^{\lfloor d/2 \rfloor})$ in $\mathbb{R}^d$

Approx: Compute a subset $P' \subseteq P$ s.t. $\mathrm{conv}(P)$ and $\mathrm{conv}(P')$ are very similar

Exact

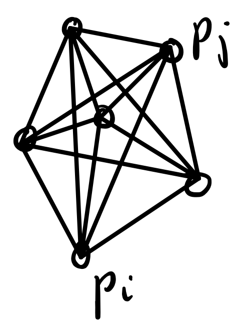


Approx



Well-Separated Pair Decomposition:

P:

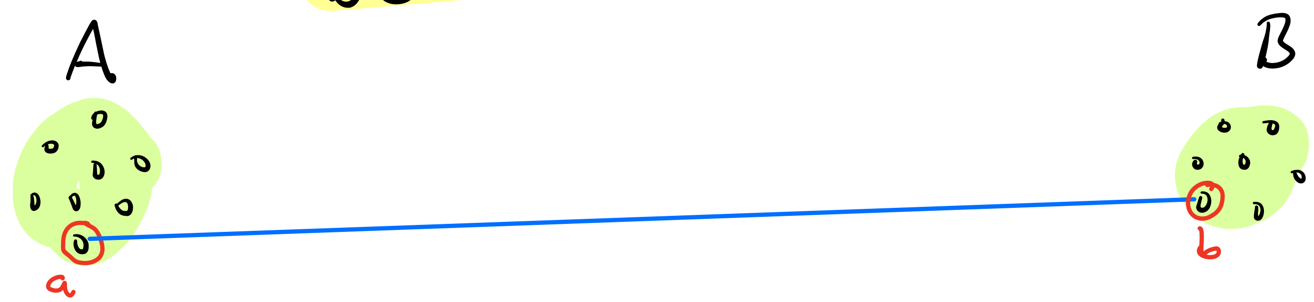


Given set $P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$, the **Euclidean graph** is complete graph on P, where $w(p_i, p_j) = \|p_i - p_j\|$

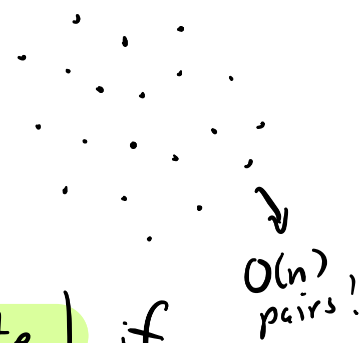
- Has $\binom{n}{2} = O(n^2)$ edges

- Can we encode this using a structure of size $O(n)$?

Intuition: If two point clusters $A, B \subseteq P$ well separated, we can represent many edges of $A \times B$ using a single edge connecting a representative $a \in A$ + $b \in B$

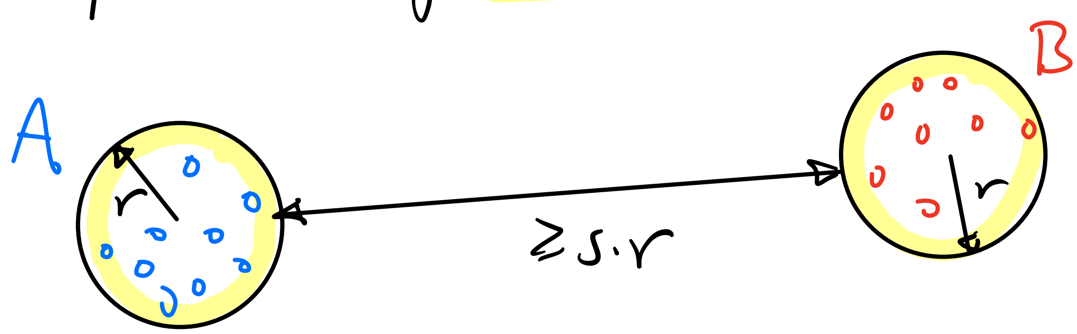


If we do this for all well-separated clusters, how many edges do we need?



Def: Given $P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$ and scalar $s > 0$

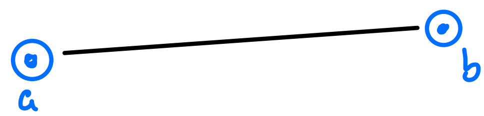
- Two sets $A, B \subseteq P$ are s-well separated if A + B can be enclosed in balls of some radius r, s.t. these balls are separated by distance $\geq s \cdot r$



Obs:

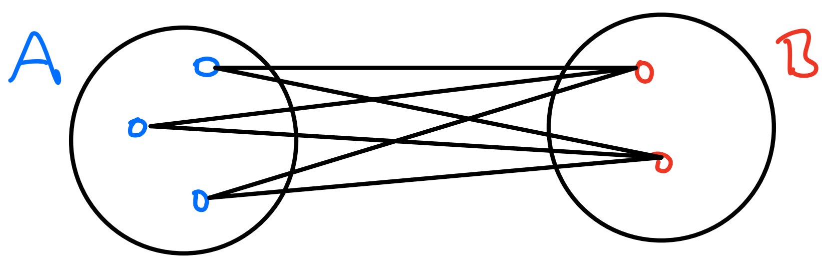
- If A + B are s-well separated, they are s'-well separated for any $0 < s' \leq s$

- Two singleton sets $A = \{a\}$, $B = \{b\}$ are s-well separated for any $s > 0$. $(a \neq b)$



Def: Given sets A, B, define

$$A \otimes B = \{\{a, b\} \mid a \in A,\ b \in B,\ a \neq b\}$$



Obs: $P \otimes P$ = set of all $\binom{n}{2}$ pairs of P.

Def: Given P + $s > 0$, an s-well separated pair decomposition of P (s-WSPD) is collection of pairs

$$\{\{A_1, B_1\}, \ldots \{A_m, B_m\}\}$$

such that:

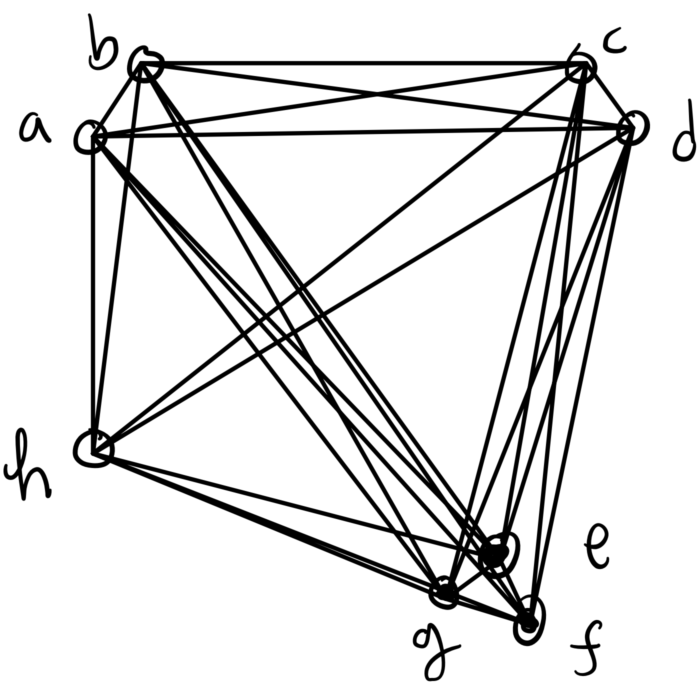
(1) $A_i, B_i \subseteq P$ for $1 \le i \le m$

(2) $A_i \cap B_i \neq \emptyset$ " " (disjoint)

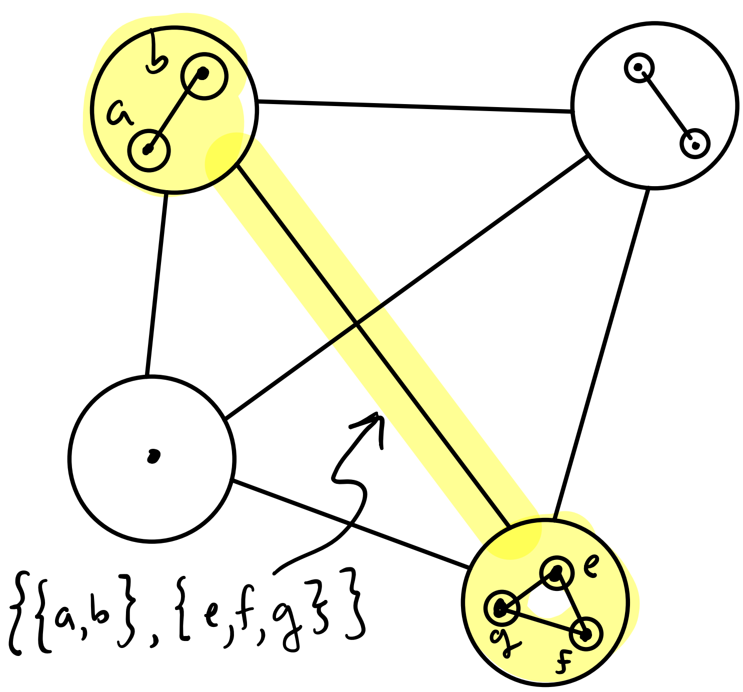
(3) $\bigcup_{i=1}^{m} A_i \otimes B_i = P \otimes P$ (cover)

(4) A_i + B_i are s-well separated for $1 \le i \le m$

28 pairs



11 well-sep pairs



Obs: For any $s > 0$ there is always a trivial s-WSPD consisting of $\binom{n}{2}$ singleton pairs.

Can we do better?

d is constant: hidden d^d

Yes! $\rightarrow$ $O(s^d \cdot n)$ pairs

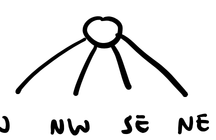
If s, d constants: $O(n)$!

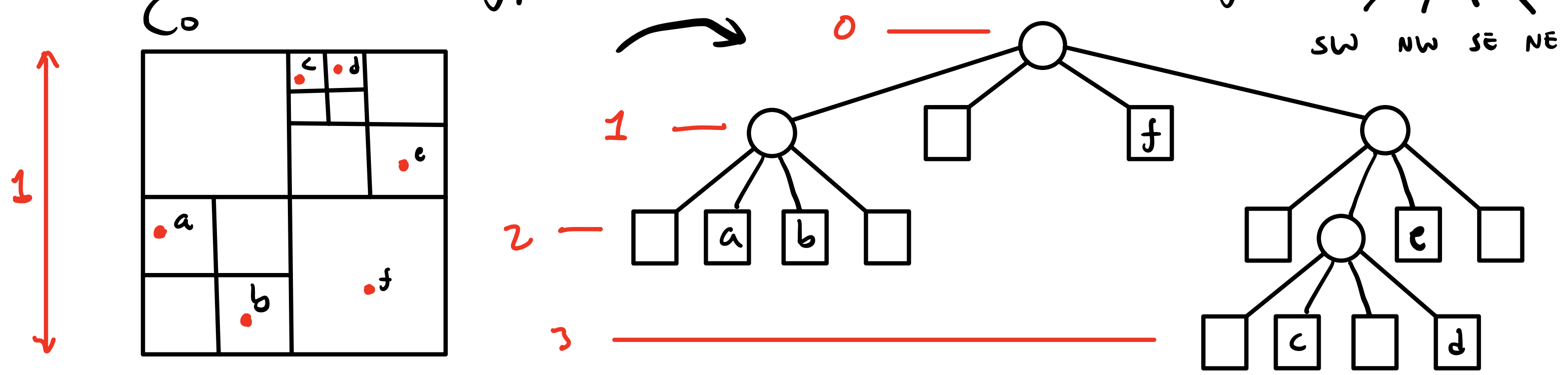
Can compute in time: $O(n \log n + s^d n)$

Quadtrees:

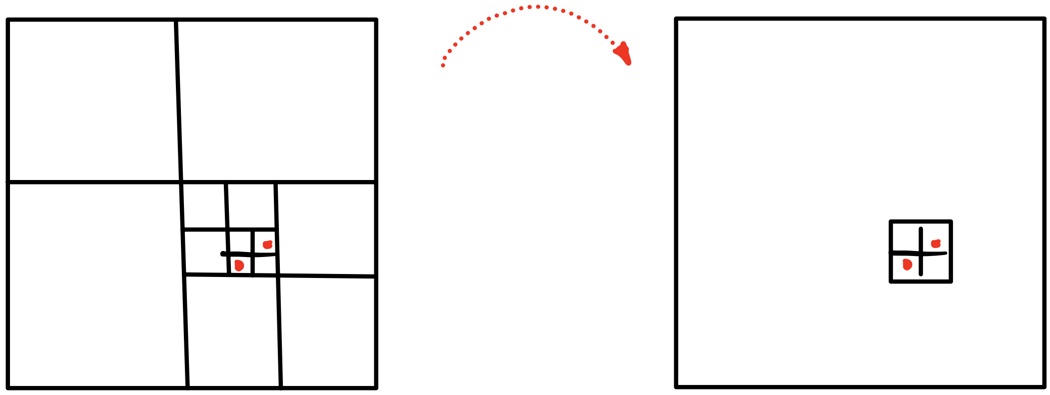
A tree storing P based on recursive subdiv. into hypercubes.

- Let C_0 be a bounding hypercube for P
- While a cell of subdivision has 2 or more pts of P, split it into 2^d hypercubes of half side length





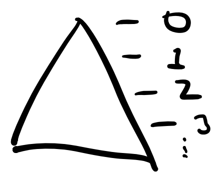
Note: A quadtree may have more than $O(n)$ nodes, but we can reduce storage to $O(n)$ by path compression. (see latex notes)



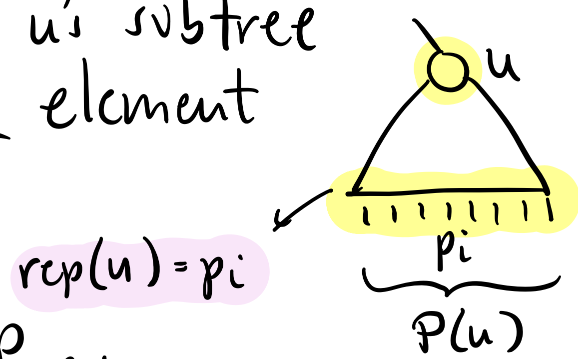
Thm: Given a set of pts $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ can construct a (compressed) quadtree of space $O(n)$ in $O(n \log n)$ time.

Additional information (provided by construction)

Given node u in tree:



- level(u) = level of u in tree
- $P(u)$ = set of pts in u's subtree
- rep(u) = an arbitrary element of $P(u)$



We will represent each WSP as pair of nodes $\{u, v\}$. Actual pair is $\{P(u), P(v)\}$

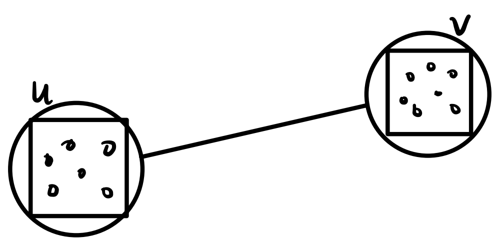
Constructing the WSPD:

Given P + $s > 0$:

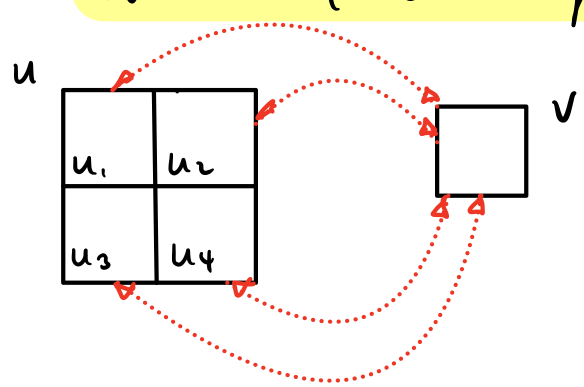
- Build quadtree for P → Let u_0 = root
- Invoke: ws-pairs(u_0, u_0, s)

```
ws-pairs(Node u, Node v, Scalar s) {
    if(u + v are both leaves + u==v) return ∅
  ← if(rep(u) or rep(v) is empty) return ∅
    else if (u + v are s-well sep)
        return {u,v}        // wsp = {P(u), P(v)}
    else    // not w.s.
        if (level(u) > level(v))
            swap u ↔ v    // u is not deeper than v
        let u_1, ..., u_k be u's children
        return ⋃_{i=1}^{k} ws-pairs(u_i, v, s)
}
```

Cases: u + v are well sep



u + v not well-sep



Analysis: We'll show $O(s^d \cdot n)$ pairs generated

- **Assume:** Quadtree is not compressed (simpler)
  $s \geq 1$ (else just use $s' = \max(1, s)$)

① **Terminal / Non-Terminal:**
- To count no. of WSP's, we'll count no. of calls to ws-pairs
- A call is:
  - terminal: makes no recursive calls
  - non-terminal: otherwise
- It suffices to count just no. of non-terminal calls (each generates at most $2^d = O(1)$ term. calls)

② **Charging:** We'll count no. of non-term calls by charging each to node of tree.

Preview:
- Each node receives $O(s^d)$ charges
- $O(n)$ nodes in tree

$\Rightarrow O(s^d \cdot n)$ total charges

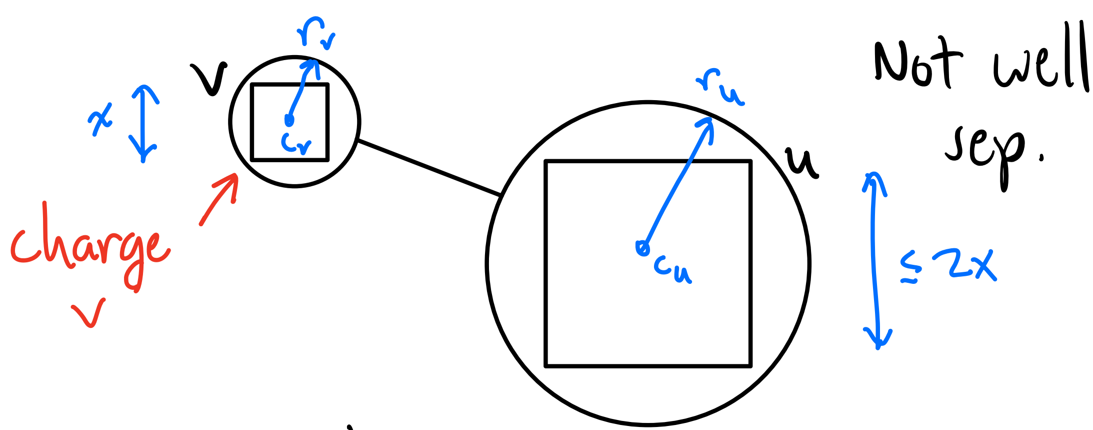
③ Who gets charged?

Let ws-pairs(u, v, s) be non-term call

$\Rightarrow$ u, v not well sep.

→ Assume (w.l.o.g.) $lev(u) \leq lev(v)$

→ We will charge v
(smaller node is charged)



- Let x be side length of v's cell
- We always split larger cell first
  $\Rightarrow$ u's side length $\leq 2x$

- Let r_v = radius of ball enclosing v's cell
  + r_u = " " " " u's cell

  $\Rightarrow$ $r_u \leq 2 r_v$

  and $r_v = x\sqrt{d}/2$
- Let c_u, c_v be centers of u + v's cells

This call is non-term

$\Rightarrow$ u, v not well separated

$\Rightarrow$ Distance between balls is

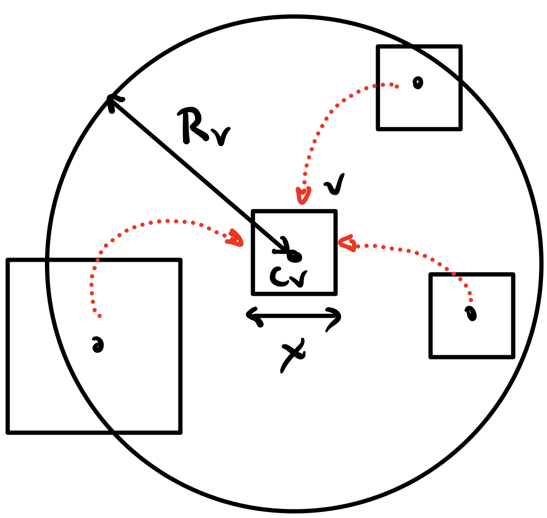
$$< s \cdot \max(r_u, r_v) \le s \cdot r_u \le s(2 \cdot r_v)$$
$$= s \cdot x \cdot \sqrt{d}$$

$\Rightarrow$ Distance between centers

$$\|c_u - c_v\| \le r_v + r_u + s x \sqrt{d}$$
$$\le x\sqrt{d}/2 + x\sqrt{d} + s x \sqrt{d}$$
$$= \left(\tfrac{1}{2} + 1 + s\right) x \sqrt{d}$$
$$< 3 s x \sqrt{d} \quad (\text{since } s \ge 1)$$

Def: $R_v = 3 s x \sqrt{d}$

Summary: A node v of side length x is charged by nodes u of side length x or $2x$ whose cell centers lie within a ball of radius $R_v = 3 s x \sqrt{d}$ of c_v.



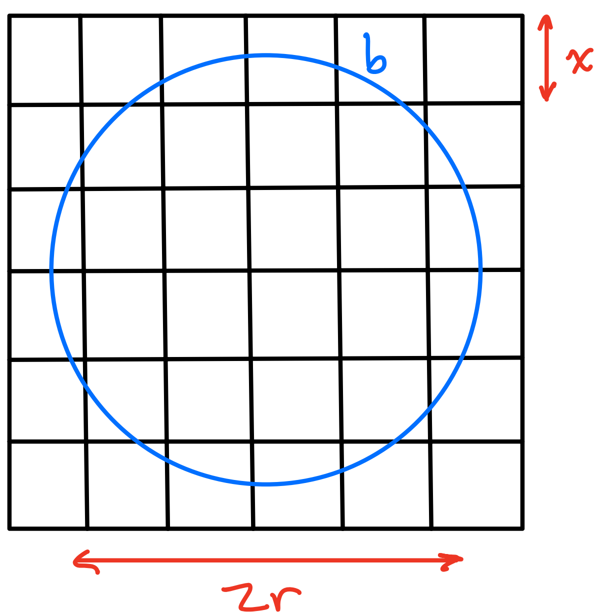
How many such nodes can there be?

Packing Lemma: Given a ball b of radius r in $\mathbb{R}^d$ + any collection X of disjoint quadtree cells of side length $\geq x$ that overlap b, then

$$|X| \leq \left(1 + \left\lceil \frac{2r}{x} \right\rceil\right)^d \leq O\left(\max\left(2, \frac{r}{x}\right)^d\right)$$

Proof: To maximize no. of cells, assume they are as small as possible $\Rightarrow x$

These cells form a grid of side length x that overlaps b



No. of grid squares of side length x overlapping an interval of length $2r$ is

$$\leq 1 + \left\lceil \frac{2r}{x} \right\rceil$$

$\Rightarrow$ Total: $\left(1 + \left\lceil \frac{2r}{x} \right\rceil\right)^d$

$\square$

Returning to WSPD analysis:

- No. of charges to v $\leq$

  No. of nodes of side length $\geq x$ overlapping a ball of radius $R_v = 3sx\sqrt{d}$

- By Packing Lemma, no. of nodes

$$\leq \left(1 + \left\lceil \frac{2R_v}{x} \right\rceil\right)^d$$

$$\leq \left(1 + \left\lceil \frac{6sx\sqrt{d}}{x} \right\rceil\right)^d$$

$$\leq \left(2 + 6s\sqrt{d}\right)^d$$

$$\leq O(s^d)$$ since $s \geq 1$, d is constant

So, each node charged $O(s^d)$ times

→ $O(n)$ nodes in quadtree

→ $O(n \cdot s^d)$ non-term calls to ws-pairs

→ $O(n \cdot s^d)$ pairs generated

whew!!

Theorem: Given a point set $P = \{p_1, \dots, p_n\}$ in $\mathbb{R}^d$ (d is constant) and $s \geq 1$, in $O(n \log n + s^d n)$ time, can build an s-WSPD for P of size $O(s^d \cdot n)$