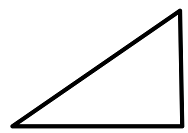
# CMSC 754 - Computational Geometry
## Lecture 20 - Motion Planning

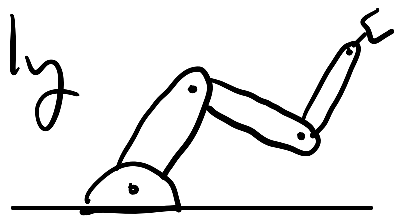
**Motion Planning:**

Given a robot (with constraints on how it can move), a set of obstacles, and a start + target configurations for the robot, is there a collision-free motion plan?

**Robot:** May be rigid object



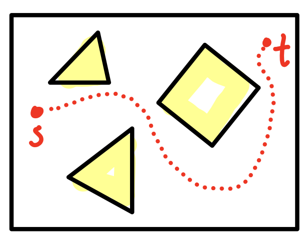
or linked/hinged assembly



**Motion constraints:**

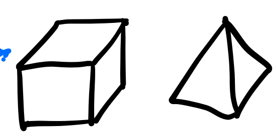
- Translation
- Rotation
- Speed/Acceleration limits

⋮



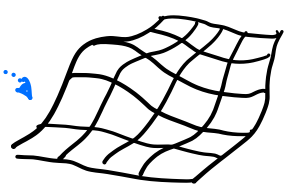

**Obstacles:** Polygons in 2-D

Polyhedra in 3-D



Curved objects

Terrains



We'll mostly consider the simplest scenario:

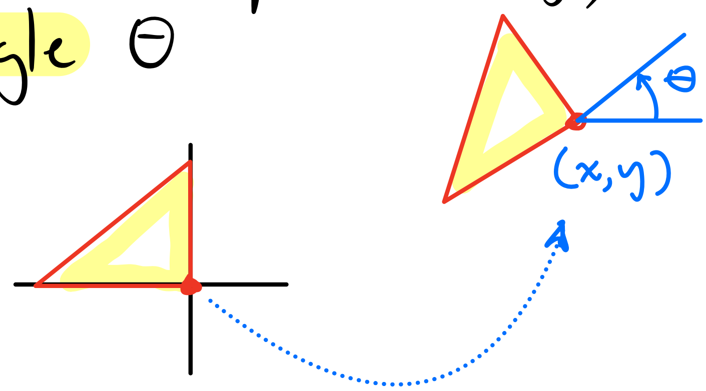
- **Space** – $\mathbb{R}^2$
- **Robot** – (convex) polygon
- **Motion** – translation only
- **Obstacles** – collection of nonoverlapping convex polygons

**Configuration:** A set of parameters that uniquely specifies the robot's position

E.g. **Rigid in 2-D**

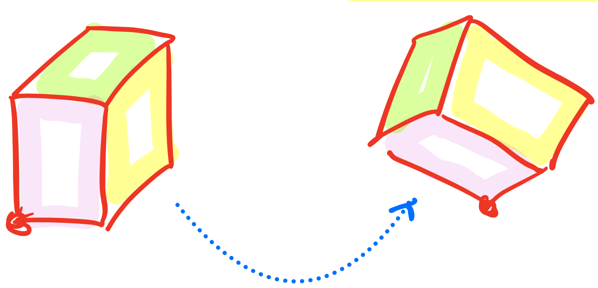
- location of reference point $(x,y)$
- rotation angle $\theta$

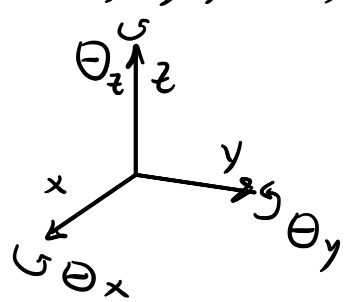
Reference position:




**Rigid in 3-D**

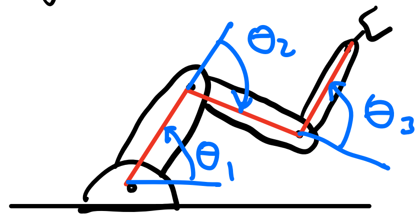
- location $(x,y,z)$
- rotation
  - Euler angles $(\theta_x, \theta_y, \theta_z)$
  - Quaternion





## Linked/Hinged: Joint angles

$(\theta_1, \theta_2, \theta_3)$




## Motion Planning in Config. Space:

- Rather than moving a robot amidst obstacles

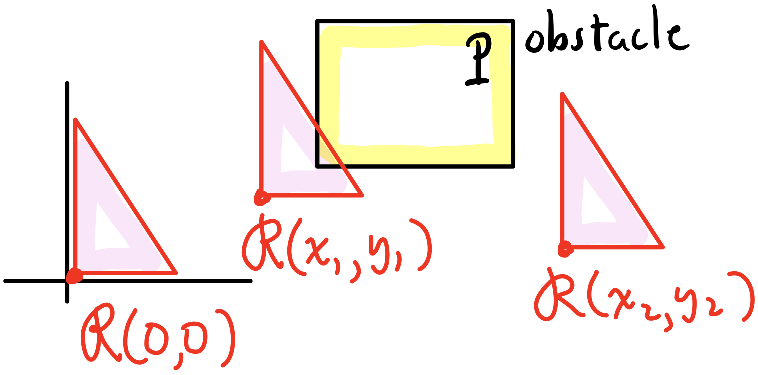
  – instead –

- Move a point in the robot's configuration space

Need to distinguish between:

free configuration – robot does not collide
forbidden configuration – robot collides

E.g. Translation only (configuration = location of ref. point)




$(x_1, y_1)$ – forbidden

$\mathcal{R}(x_1, y_1) \cap \mathcal{P} \neq \emptyset$

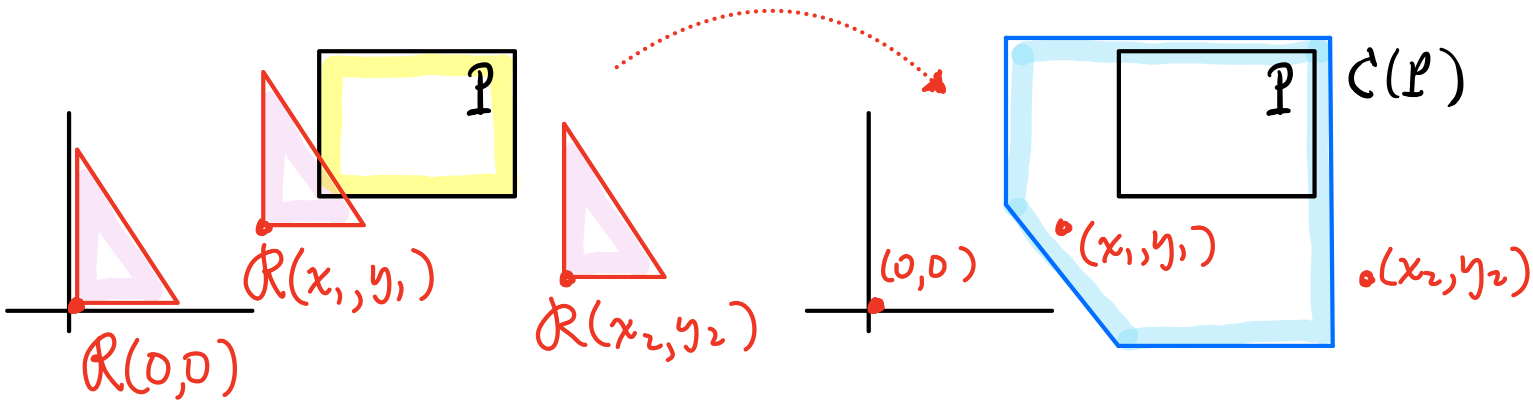
$(x_2, y_2)$ – free

$\mathcal{R}(x_2, y_2) \cap \mathcal{P} = \emptyset$

## Configuration Obstacle (or C-Obstacle)

Given robot $\mathcal{R}$, config vector $v$, obstacle $P$ the C-obstacle for $P$ is:

$$C_{\mathcal{R}}(P) = \{ v \mid \mathcal{R}(v) \cap P \neq \emptyset \}$$




## C-Obstacles for Translation - Minkowski Sum

The easiest C-obstacles are for translational motion.

**Def:** Given $P, Q, S \subseteq \mathbb{R}^d$ + $\alpha \in \mathbb{R}$

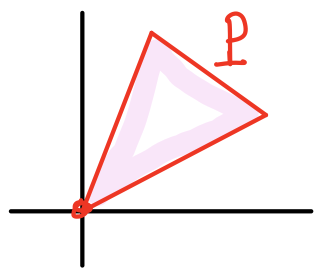
$$P \oplus Q = \{ p + q : p \in P, q \in Q \}$$

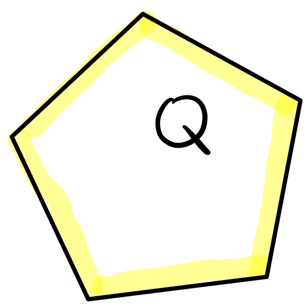
Minkowski Sum

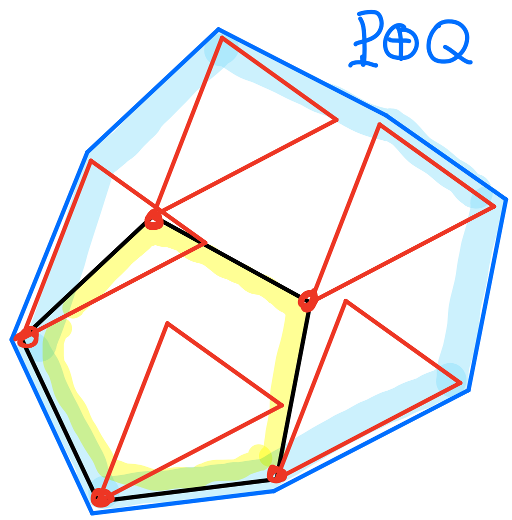
$$\alpha P = \{ \alpha \cdot p : p \in P \}$$

$$-P = \{ -p : p \in P \}$$

**Intuition:** $P \oplus Q$ - Place $P$ so its ref. pt. is at origin
- Sweep $P$'s ref pt around $Q$ + see what's swept out



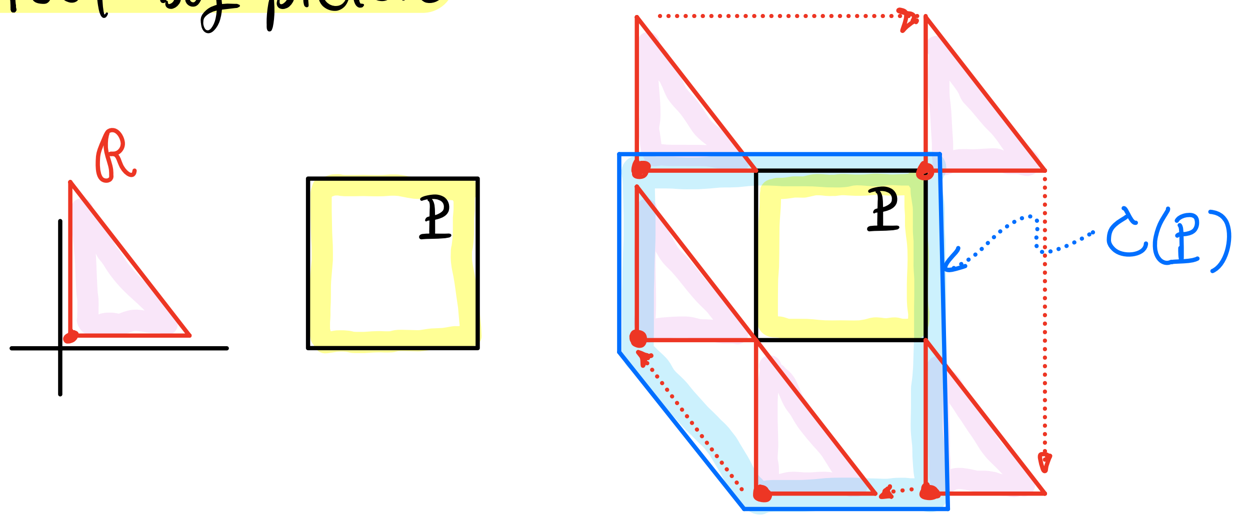

**Lemma:** Given a translating robot $R$ + obstacle $P$:

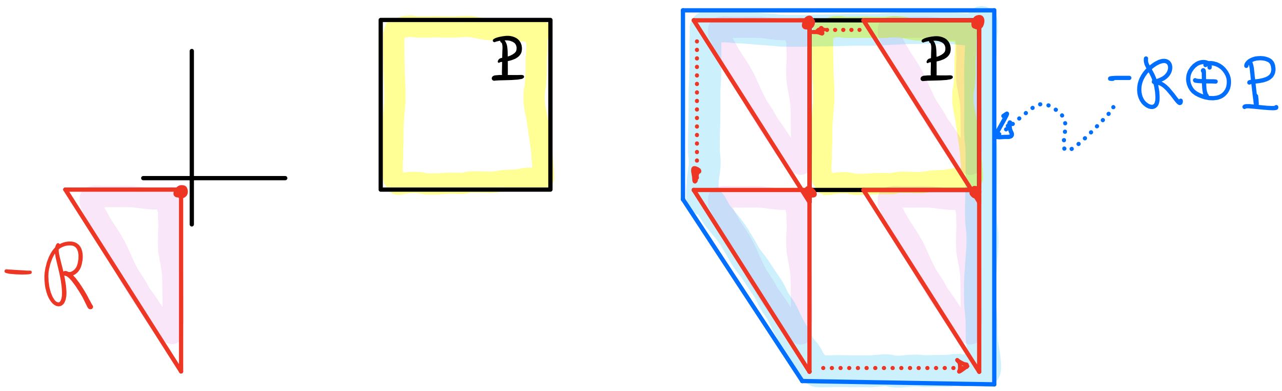
$$C_R(P) = P \oplus (-R)$$

**Proof:** For any translation vector $t$

$$
\begin{aligned}
t \in C(P) &\iff R(t) \text{ collides with } P \\
&\iff R+t \cap P \neq \emptyset \\
&\iff \exists\, r \in R,\ p \in P \quad r+t = p \\
&\iff \quad \text{"} \qquad \text{"} \qquad t = p - r \\
&\iff t \in P \oplus (-R)
\end{aligned}
$$

## Proof by picture:

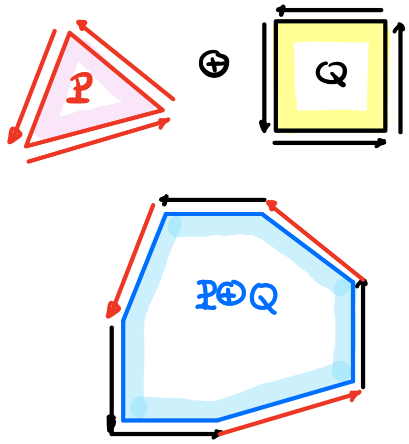



## Computing the Minkowski Sum:

If $P$ is a convex $m$-gon
$Q$ is a convex $n$-gon
can compute $P \oplus Q$ in time $O(m+n)$

- Direct edges CCW (vectors)
- Sort them by angle
- Join them tail to head



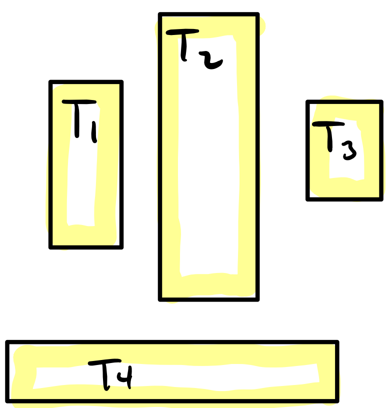
# Complexity of C-Obstacles:

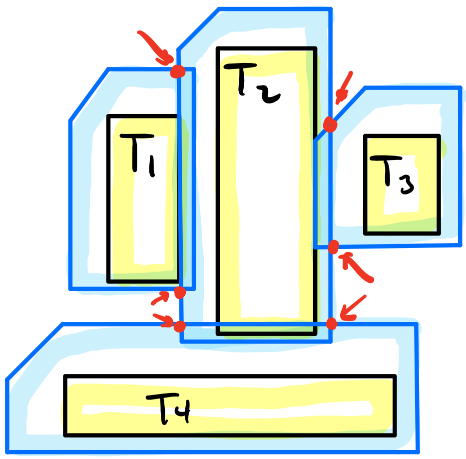
- Suppose we have an m-sided convex robot $R$ and a collection of disjoint convex obstacles $T_1, \ldots, T_k$. Let $n_i$ = num. of sides in $T_i$. Let $n = \sum n_i$

- What is total size of config. obstacles?

$$\bigcup_{i=1}^{k} C_R(T_i) = \bigcup_{i=1}^{k} T_i \oplus (-R)$$

- Although $T_i$'s are disjoint, $C_R(T_i)$ may overlap

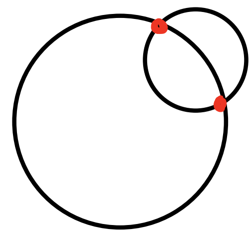


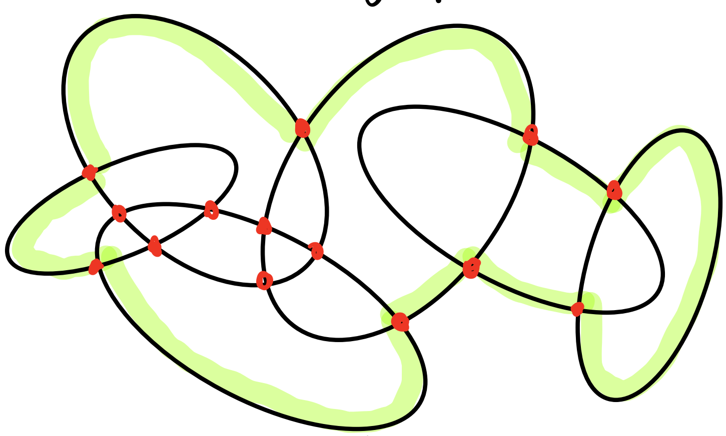

- Points of boundary overlaps create additional vertices - How many? $O(n)$ $O(n^2)$?

## Pseudodisks:

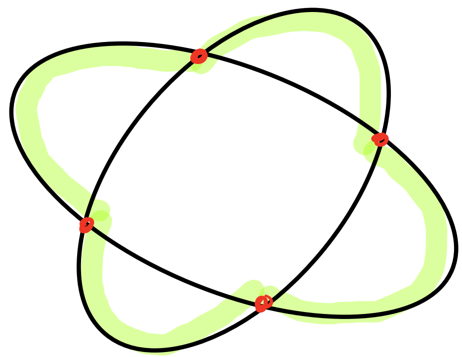
- The boundaries of two circular disks intersect at most twice.



- A collection of convex objects $\{o_1, \dots, o_k\}$ is a collection of pseudodisks if the boundaries of any pair intersect at most twice.



Collection of pseudodisks



Not pseudodisks

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**Lemma:** Given a set $T_1, \dots, T_k$ of disjoint convex bodies in $\mathbb{R}^2$ and convex $R$

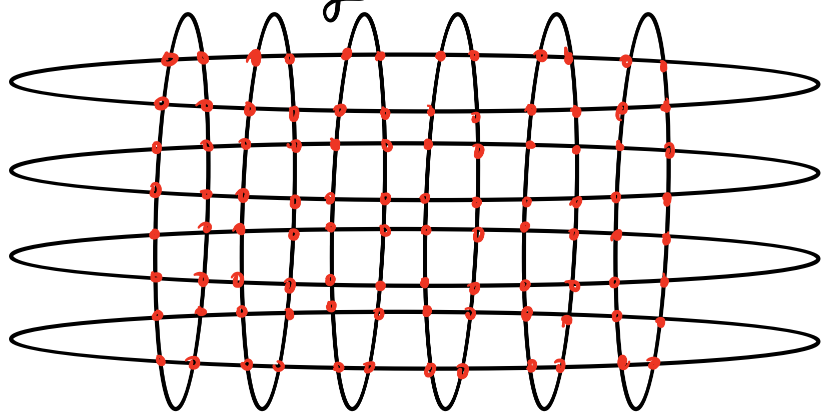
$$\{C_R(T_1), \dots, C_R(T_k)\} = \{T_1 \oplus (-R), \dots, T_k \oplus (-R)\}$$

is a collection of pseudodisks.

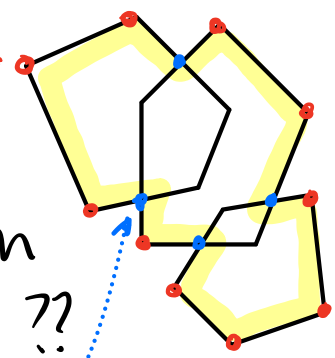
**Proof:** See latex notes

> **Theorem:** Given a collection of pseudodisks with a total of $n$ vertices, their union has a total of $O(n)$ vertices.

In general, union may have $O(n^2)$ vertices

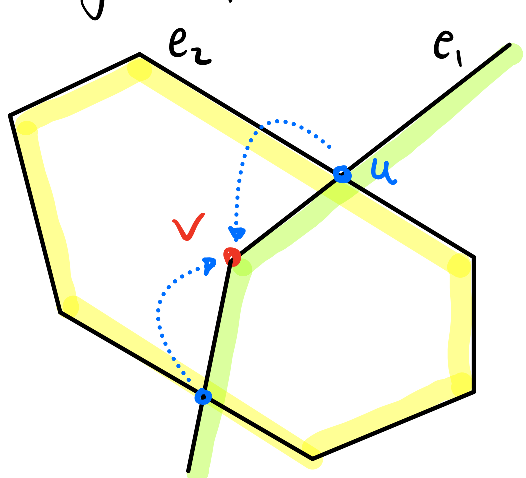


Vertex types:



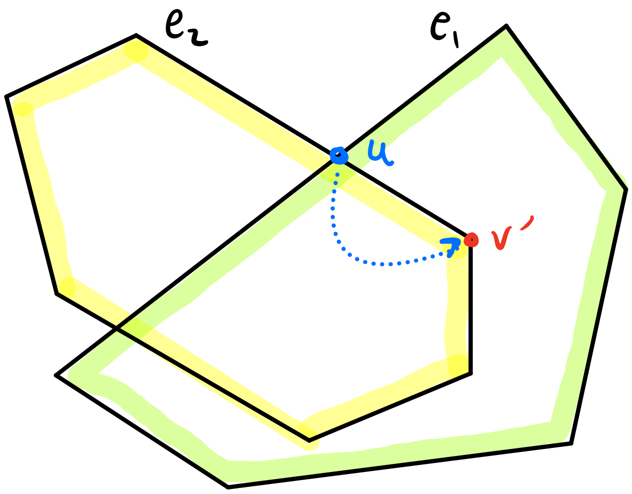
- Vertices of original polygons – $n$
- Vertices caused by intersections – ??

- We'll "charge" intersection vertices to vertices hidden in the interior
- Suppose edges $e_1$ + $e_2$ intersect at $u$



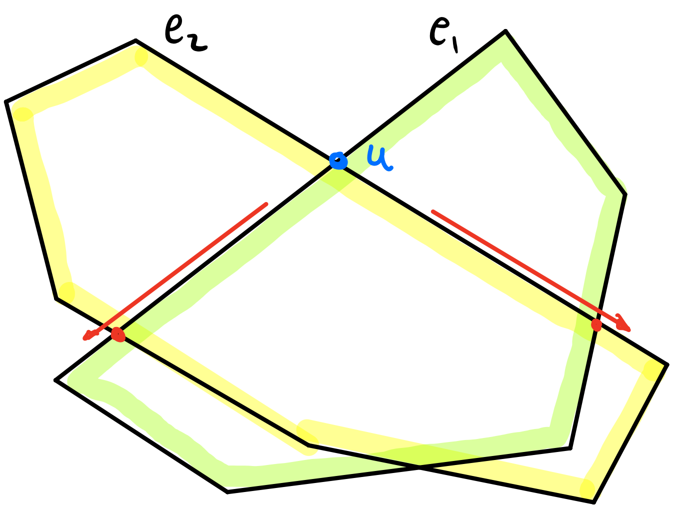

- if $e_1$ leads to internal vertex $v$, charge $u$ to $v$
- $v$ gets $\leq 2$ charges

- Otherwise, if $e_1$ cuts through, but $e_2$ leads to internal vertex $v'$, charge $u$ to $v'$




(Again $v'$ can be charged at most twice)

- Otherwise both $e_1$ + $e_2$ cut through the other polygon




But this cannot happen since these are pseudodisks!

Since every vertex is charged at most twice union has at most $2n$ vertices. □

**Theorem:** Given a convex m-sided robot and and a collection of n disjoint obstacles, each with $O(1)$ sides, the total boundary complexity of the union of C-obstacles is $O(m \cdot n)$

**Proof:** We have a collection of n pseudodisks each with $O(1) + m = O(m)$ sides.

$\Rightarrow$ Total vertices is $O(m \cdot n)$

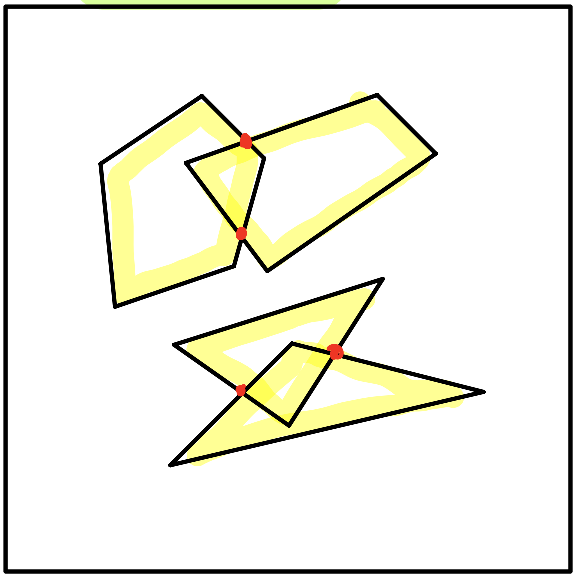
$\Rightarrow$ Union complexity is $O(m \cdot n)$.

## Path Planning in Config Space:

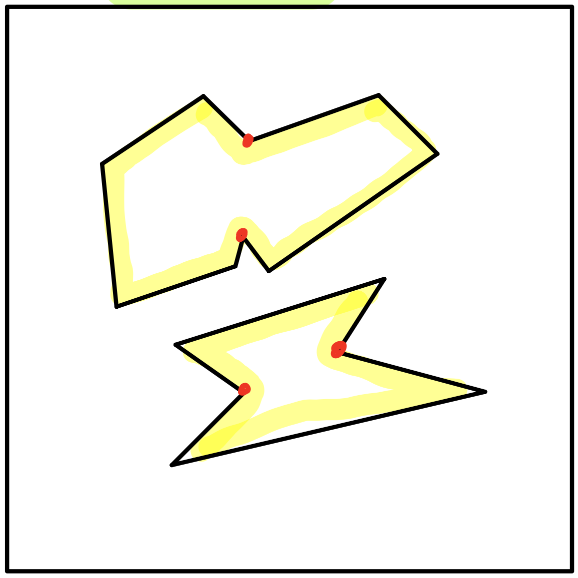
Once we have computed the C-obstacles, how to find a path between start + target?

- Compute union of C-obstacles
- Compute a decomposition of the compliment space (outside the C-obstacles)
  Eg. Triangulate or trapezoid map
- Compute dual graph, joining pairs that can reach each other

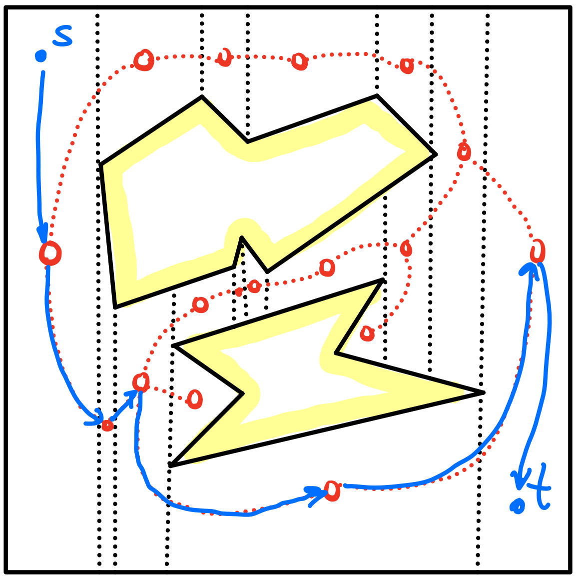
C-obst.



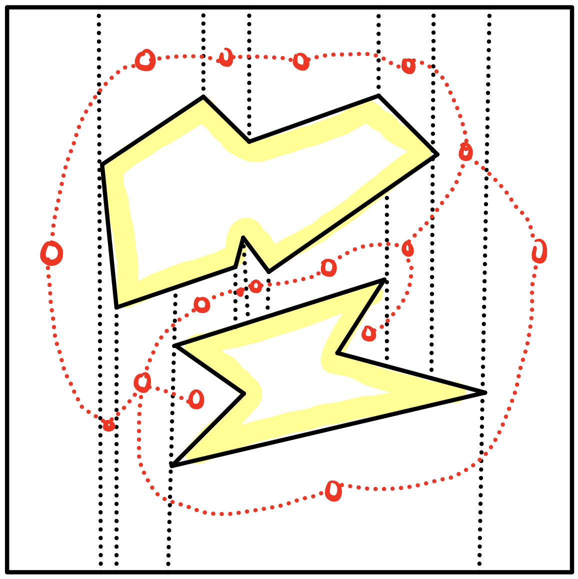
Union



Path $s \rightsquigarrow t$




Trap. Map



**Finally**: Given start $s$ + target $t$,
- find trapezoids containing them
- if reachable in dual graph
  - create path joining them
- else - output "unreachable"

Note: Not the shortest path