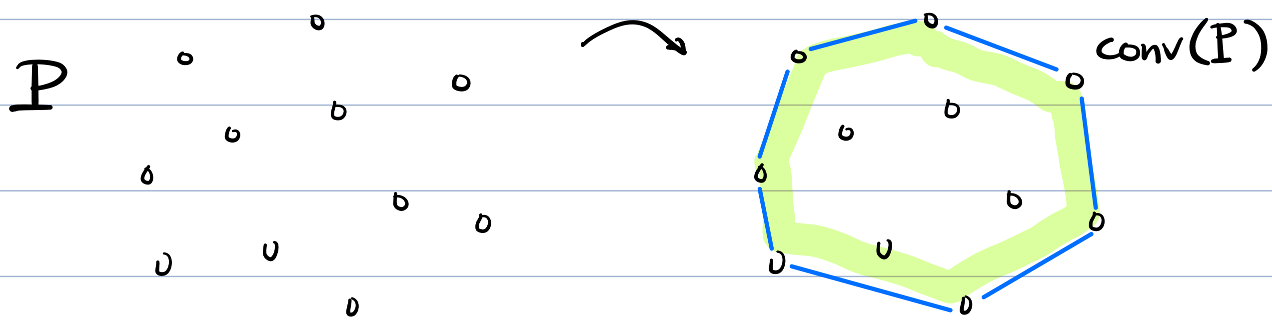
CMSC 754 - Computational Geometry
Lecture 2: Convex Hulls (in the plane)

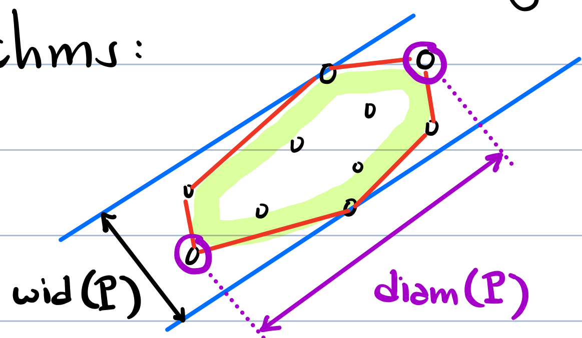
Convex Hull (intuitive definition)

Given a point set P in $\mathbb{R}^2$, imagine snapping a rubber band around the points:



Uses:

- shape approximation (for intersection testing)
- first step in other algorithms:
  - diameter
  - width



Geometric Basics:

Affine Geometry - The geometry of "flats"

Concepts:

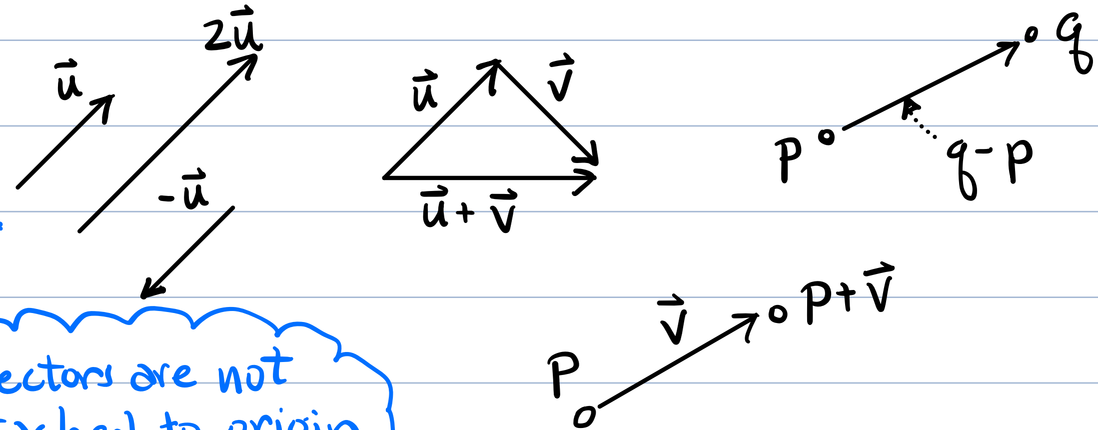
- Scalars (Real numbers) $3, -17, \pi, \sqrt{2}, \ldots$
- Points (represent positions) $\cdot p \quad \cdot q$
- Free vectors (direction + magnitude) (not anchored to the origin) $\vec{u}$ $\vec{v}$

Operations:

s = scalar ; V = vector ; P = point

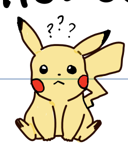
- $s \cdot V \to V$ - scalar-vector mult
- $V + V \to V$ - vector addition (+ subtraction)
- $P - P \to V$ - point subtraction (result is vect!)
- $P + V \to P$ - point-vector addition

E.g.



Vectors are not attached to origin

Note: $P + P$ is not allowed!



Affine Combinations:

Can be negative!

Given points p + q and scalars $\alpha + \beta$ where
$\alpha + \beta = 1$

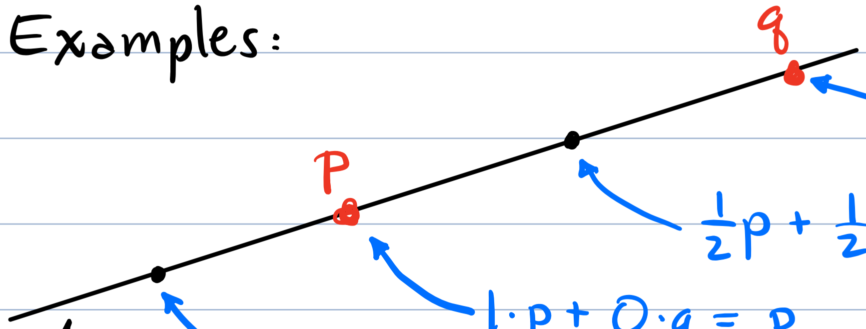
Def:

$$\mathrm{aff}(p, q; \alpha, \beta) := p + \beta(q - p)$$

Equivalent to: $p + \beta q - \beta p$
$= (1-\beta)p + \beta q$
$= \alpha p + \beta q$

Weighted sum of p+q

Examples:



$0 \cdot p + 1 \cdot q = q$

$\frac{1}{2}p + \frac{1}{2}q$ = midpoint

$1 \cdot p + 0 \cdot q = p$

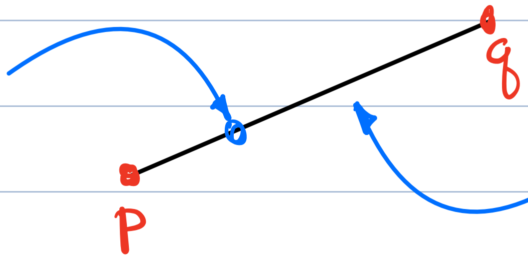
$\frac{3}{2}p - \frac{1}{2}q$

line $\overleftrightarrow{pq} = \{ \text{aff}(p,q) \mid \forall \alpha, \beta \;\; \alpha + \beta = 1 \}$

Convex Combination:

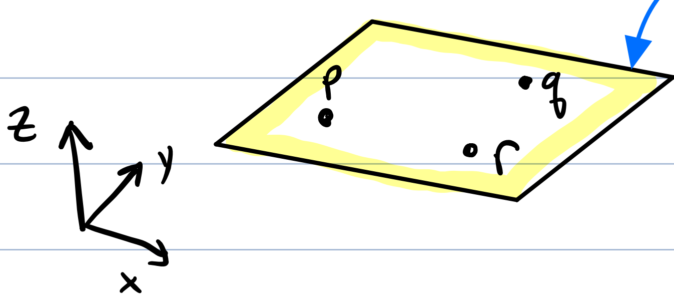
Special case when $0 \le \alpha, \beta \le 1$

$\frac{2}{3}p + \frac{1}{3}q$

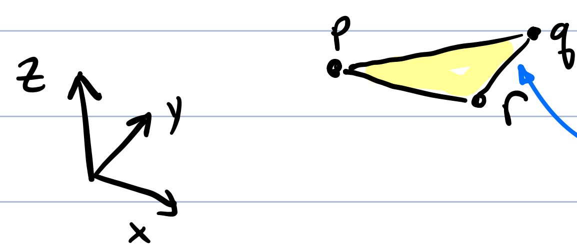


segment $\overline{pq} =$
$\{ \alpha p + \beta q \mid \alpha + \beta = 1$
and $0 \le \alpha, \beta \le 1 \}$

Affine + convex combinations generalize to multiple points: $\sum_i \alpha_i p_i$ s.t. $\sum_i \alpha_i = 1$ (affine)
$+ \; 0 \le \alpha_i \le 1$ (convex)



Affine closure of p, q, r in $\mathbb{R}^3$ is the plane through these points



Convex closure is the triangle Δpqr

... but Affine geometry is limited–
no means for defining lengths or angles

Euclidean Geometry rectifies this by adding...

Inner product – A linear operator that maps two vectors to a scalar

$$\langle V, V \rangle \to S$$

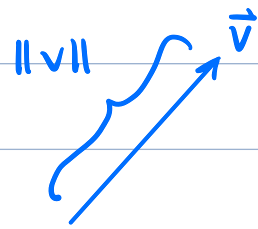
Many varieties, but most common is...

Dot product. Given $\vec{u}, \vec{v} \in \mathbb{R}^d$ $\quad \vec{u} = (u_1, \dots, u_d)$

$$\vec{u} \cdot \vec{v} := u_1 v_1 + \cdots + u_d v_d$$
$$= \sum_{i=1}^{d} u_i v_i$$

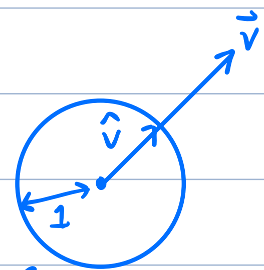
Inner product allows us to define:

Vector length: $\|\vec{v}\| := \sqrt{\vec{v} \cdot \vec{v}}$



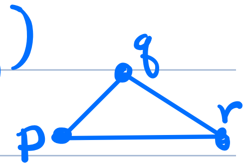
Normalization (to unit length):

$$\hat{v} := \vec{v} / \|v\| \quad (\text{for } \vec{v} \neq 0)$$



Distance: Given points p, q

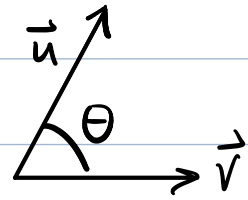
$$\text{dist}(p,q) := \|q - p\|$$



- Symmetric: $\text{dist}(p,q) = \text{dist}(q,p)$
- Triangle inequality: $\text{dist}(p,r) \leq \text{dist}(p,q) + \text{dist}(q,r)$

Angles: $\vec{u},\vec{v} \neq 0$

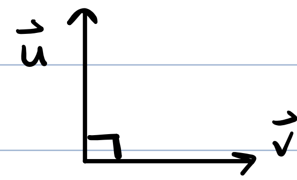
$$\operatorname{ang}(\vec{u},\vec{v}) := \cos^{-1}(\hat{u}\cdot\hat{v})$$

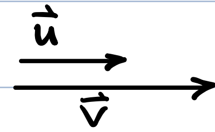


Perpendicularity / Parallelity: $\vec{u},\vec{v} \neq 0$

$$u \perp v \iff \vec{u}\cdot\vec{v} = 0$$

$$u \parallel v \iff \|\hat{u}\cdot\hat{v}\| = 1$$





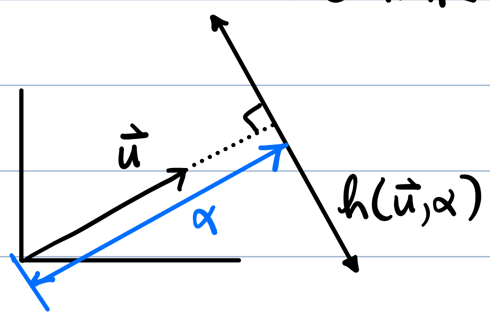
Flats (lines + hyperplanes)

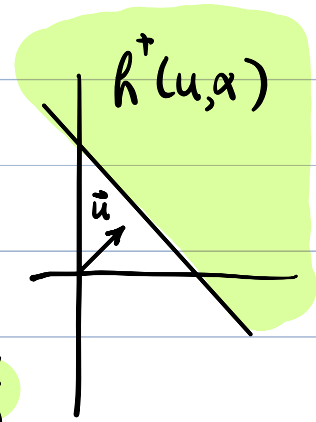
Given vector $\vec{u} \neq 0$ + scalar α, define **flat**

$$h(\vec{u},\alpha) := \{p \mid p\cdot\vec{u} = \alpha\}$$

– line in $\mathbb{R}^2$, hyperplane in $\mathbb{R}^d$

Picture
(when $\|u\| = 1$)

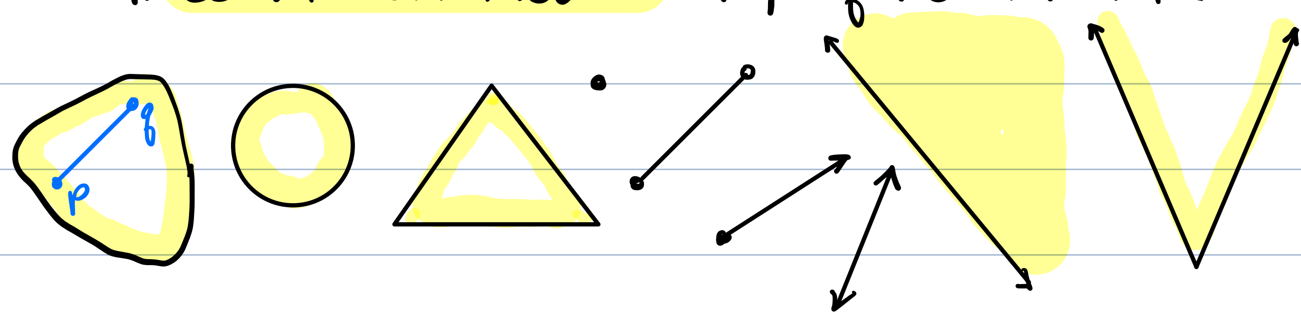


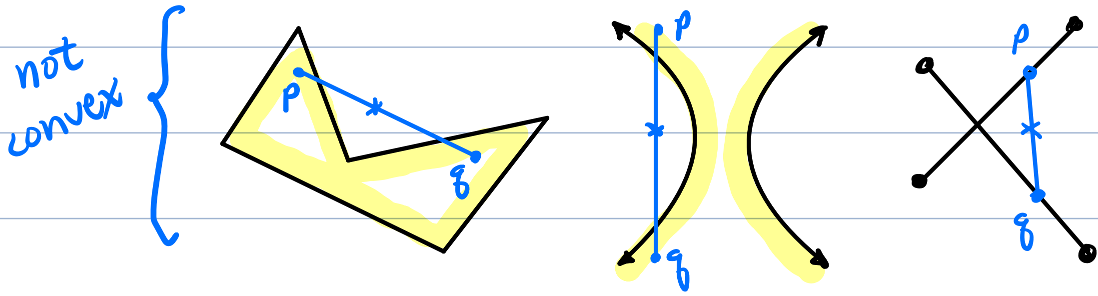


Halfspace: The region on **one side** of a **hyperplane**: $h^+(\vec{u},\alpha) := \{p \mid p\cdot\vec{u} \geq \alpha\}$

Convex: A point set $K \subseteq \mathbb{R}^d$ is **convex** if $\forall p,q \in K$, all **convex combinations** of p + q lie within K.

convex





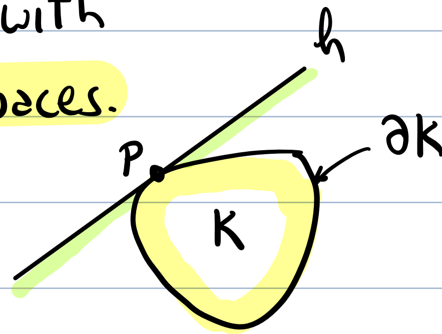
Intersection of convex sets is always convex

Union of convex sets need not be convex

Means: Boundary of K

Supporting Hyperplane:

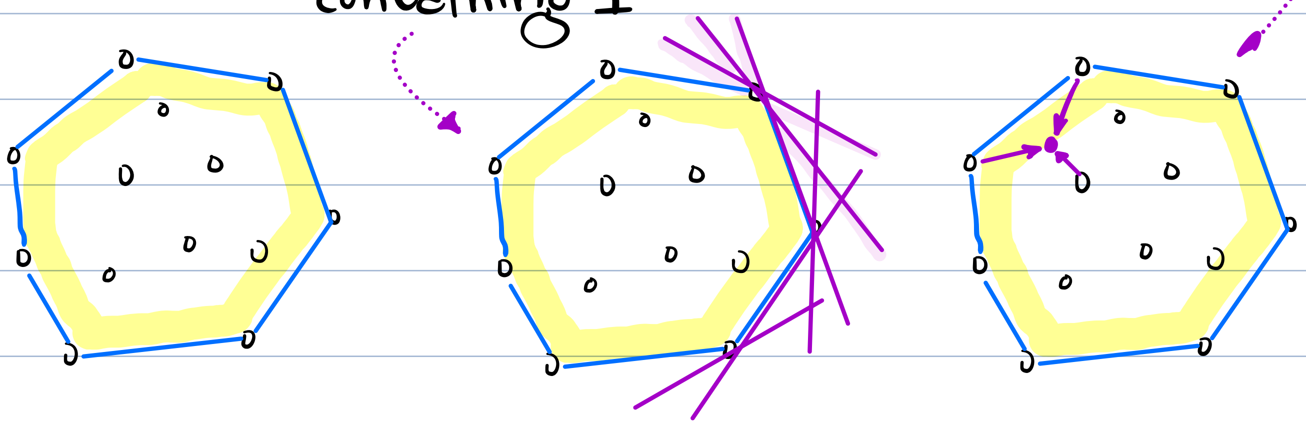
Given a convex K and any point $p \in \partial K$,
$\exists$ hyperplane h passing through p with
K lying in one of h's closed halfspaces.



Convex Hull:

Given a point set P in $\mathbb{R}^d$, the convex hull, $\text{conv}(P)$, is the smallest convex set containing P.

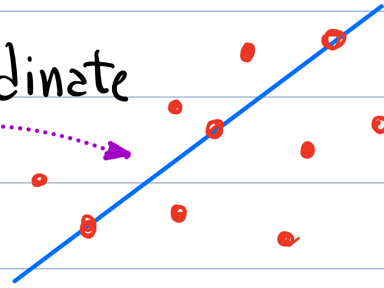
Equiv: - Set of all convex combinations of P
- Intersection of all halfspaces containing P



General Position:

Geometric algorithms are often complicated by (rare) geometric degeneracies

E.g.
- points having the same coordinate
- ≥ 3 collinear points
- ≥ 4 cocircular points



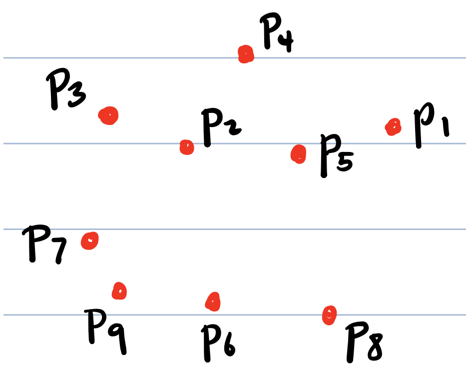
To simplify our algorithms, we assume inputs are perturbed (infitessimally) to avoid these.

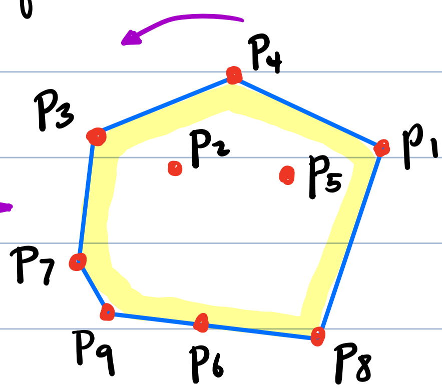
We say such inputs are in "general position"

(Planar) Convex Hull problem – Given n points

$P = \{p_1 \dots p_n\} \subseteq \mathbb{R}^2$, where $p_i = (x_i, y_i)$, compute convex hull of P, $\mathrm{conv}(P)$.

Output: Cyclic sequence of vertices on the hull





output:

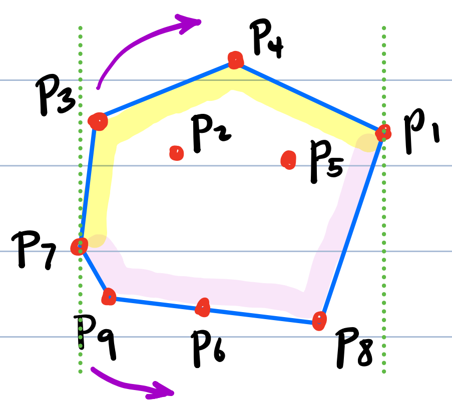
$\langle 4,3,7,9,8,1 \rangle$

Note: 6 not included. Not a vertex

Alternative output:

Upper hull + Lower hull

$\langle 7,3,4,1 \rangle + \langle 7,9,6,8,1 \rangle$



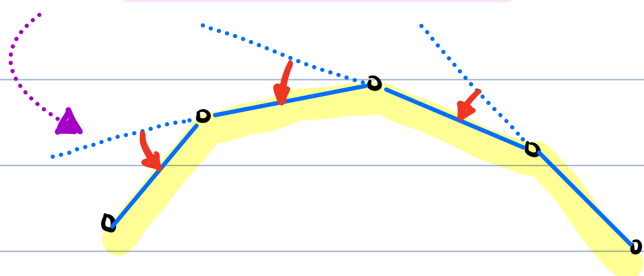
Graham's Scan - An $O(n \log n)$ algorithm

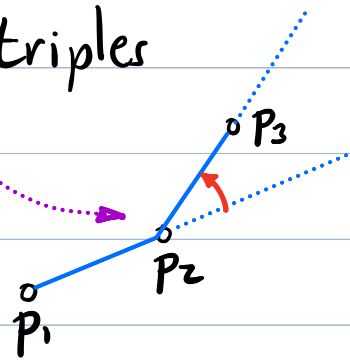
- Computes upper + lower hulls separately
- Upper hull
  - Sort points by x-coords.
  - Add each to upper hull
  - Remove points no longer on hull
- Lower hull (symmetrical)

How?

Observations:

- The rightmost point is always on the hull
- Reading right to left, consecutive triples make left-hand turns

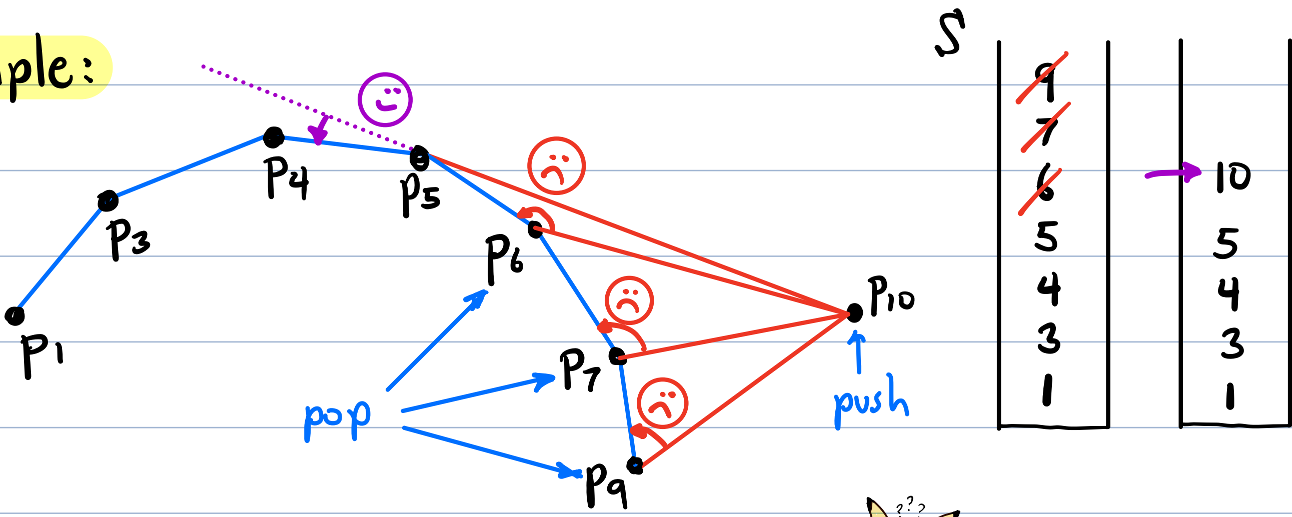




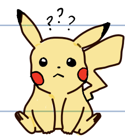
Incremental approach:

Store vertices of current upper hull on stack S
For each new point p_i (left to right)
    While $\langle p_i, S[top], S[top-1] \rangle$ do not make left turn - pop S
    Push p_i

Example:



But how to test for left turn?



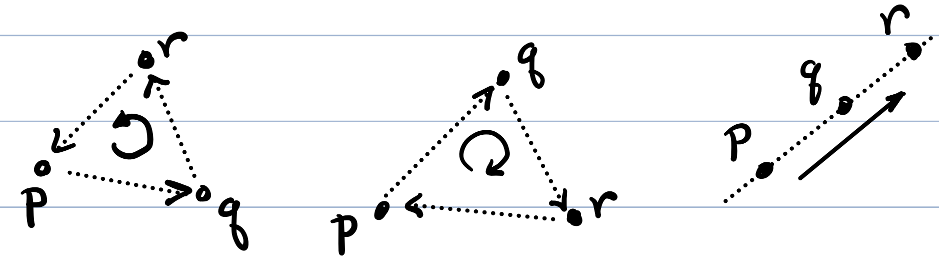
Orientation Test: For $p, q \in \mathbb{R}^1$, we have either:

	Order:		Sign:	Orient:
p q	$p < q$		$q - p > 0$	$+$
p=q	$p = q$	$\equiv$	$q - p = 0$	0
q p	$p > q$		$q - p < 0$	$-$

How to define "order" in $\mathbb{R}^2$?

Ans: **Order (orientation)** is a property of a sequence of **3 points**:

$\langle p, q, r \rangle$



Orientation:	counter-clockwise	clockwise	collinear
Sign:	$+$	$-$	0

How to compute?

$$\text{orient}(p,q,r) = \text{sign}\det\begin{bmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{bmatrix}$$

Note: (Cyclic) order of points is critical!

Higher dimensions – Orientation in $\mathbb{R}^d$ is a property of a sequence of $d+1$ points

Graham's Scan (upper hull only)

Sort points by increasing x-coords → $\langle p_1, \dots, p_n \rangle$

Push p_1 + p_2 onto S

```
for i ← 3 to n
    while (orient(p_i, S[top], S[top-1]) ≤ 0)
        pop(S)
    push p_i
```

Right turning or straight

Correctness:

$\boxed{\text{Lemma: After processing } p_i,\ S \text{ contains upper hull of } \{p_1, \dots, p_i\}}$

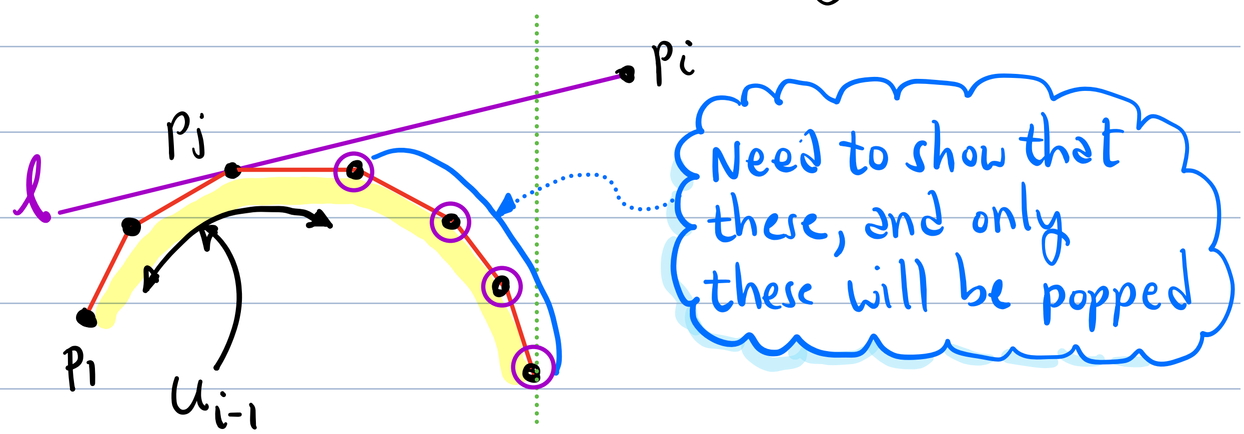
Proof: By induction on i

Basis: $i = 1, 2 \rightarrow$ Trivial

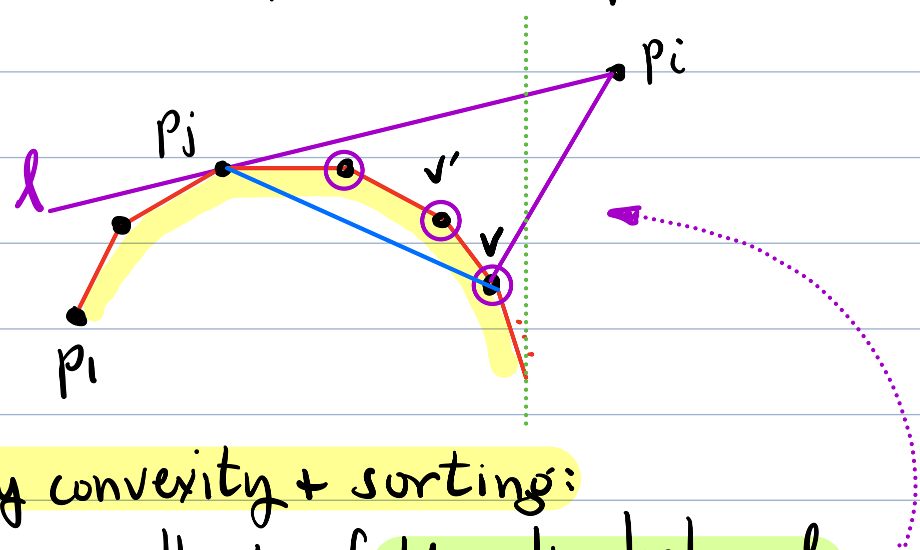
Step: For $i \geq 3$, let U_i denote vertices of upper hull of $\{p_1, \ldots, p_i\}$

- By induction U_{i-1} is correct up to p_{i-1}
- Let p_j be vertex prior to p_i on U_i

$\Rightarrow$ line $\ell := \overleftrightarrow{p_j p_i}$ is a supporting line for U_i



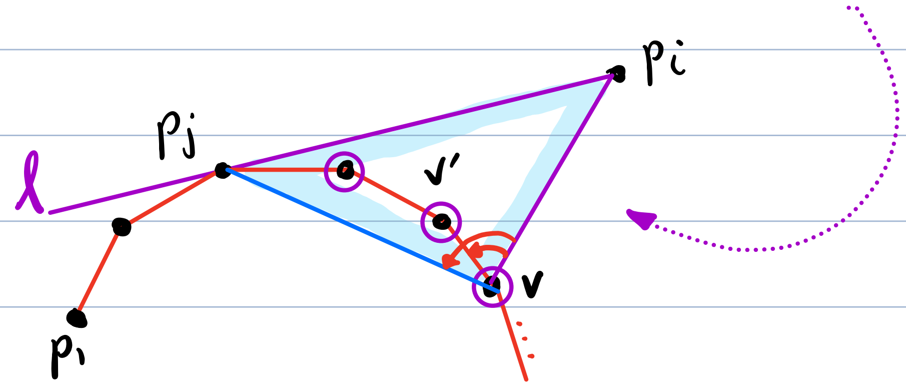
Let $v := S[top]$ + $v' := S[top-1]$



By convexity + sorting:

- all pts of U_{i-1} lie below ℓ
- all pts of U_{i-1} from p_j to v are above ℓ

$\Rightarrow$ All points of S from p_j to v lie in $\Delta p_j v p_i$



$\Rightarrow \angle p_i v v' \leq \angle p_i v p_j \leq \pi$

$\Rightarrow$ orient(p_i, v, v') = orient$(p_i, S[top], S[top-1])$
≤ 0

$\Rightarrow$ v is popped off stack

The popping repeats until $v = p_j$

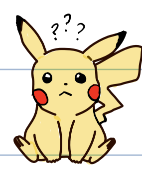
$\Rightarrow$ Final stack $S = \langle p_1, \dots, p_j, p_i \rangle = U_i$

$\square$

Running Time:

- Sort by x-coords – $O(n \log n)$
- Popping time?
    - Outer loop $i \leftarrow 3$ to n
    - Inner loop
        - Could pop up to $i-2$ points
    - Total:

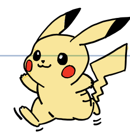
$$\leq \sum_{i=3}^{n} i-2 = O(n^2)$$



Why is this wrong?

- Each point can be popped only once.
- Let d_i = no. of points popped when p_i is processed
- $\sum_i d_i \leq n$
- Total inner loop time:

$$\sum_{i=3}^{n} 1+d_i \leq \sum_{i=3}^{n} d_i + (n-2) \leq O(n)$$



Total time: $O(n \log n) + O(n) = O(n \log n)$

Summary:

- Affine + Euclidean Geometry
- Affine + Convex combinations
- Geometric operations
  - inner product
  - orientation test
- Convexity + convex hulls
- Graham's Algorithm