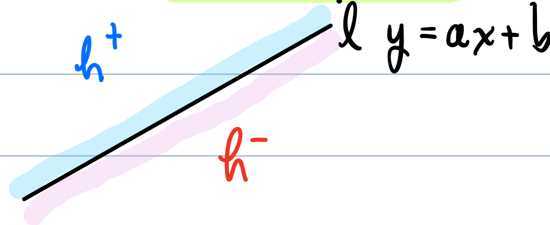
CMSC 754 – Computational Geometry
Lecture 6: Halfplane Intersection & Point-Line Duality

Halfplane Intersection:

Recall that each line in the plane defines two halfplanes



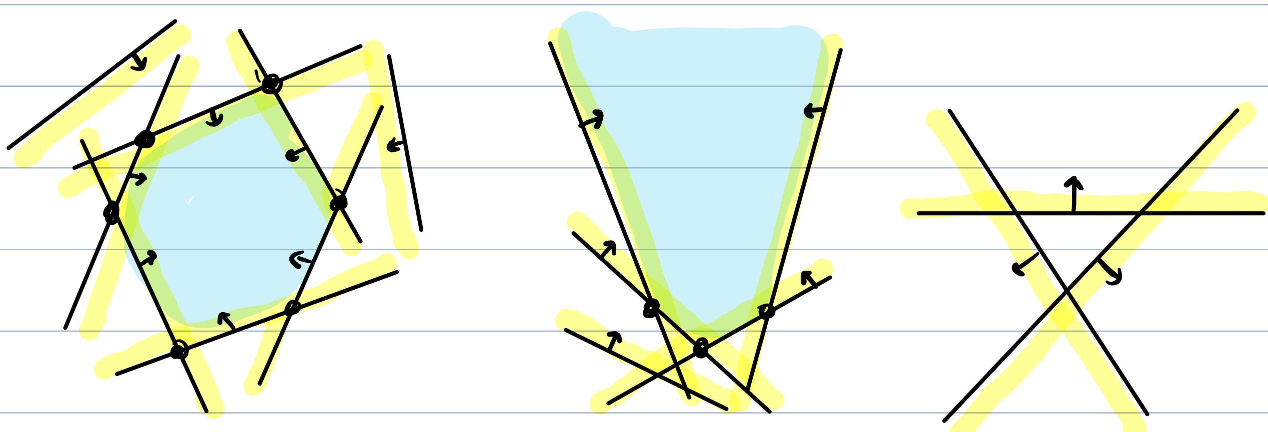
E.g.: $\ell: y = ax + b$

$h^+: y \geq ax + b$ (upper)

$h^-: y \leq ax + b$ (lower)

A halfplane is an (unbounded) convex set

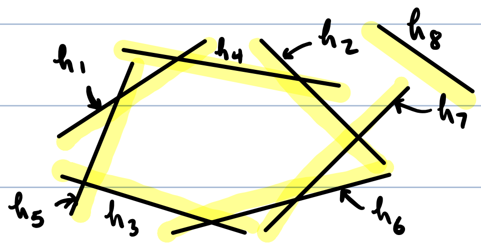
Given a set of halfplanes $H = \{h_1, \ldots, h_n\}$, their intersection $\bigcap_{i=1}^{n} h_i$ is a (possibly unbounded, possibly empty) convex polygon.



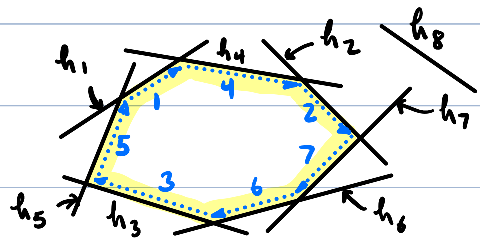
Bounded   Unbounded   Empty

Halfplane Intersection: Given halfplanes $\{h_1, \ldots, h_n\}$ compute $\bigcap_{i=1}^{n} h_i$, output as sequence of edges

Input:



Output:



⟨5, 1, 4, 2, 7, 6, 3⟩

Divide + Conquer Algorithm

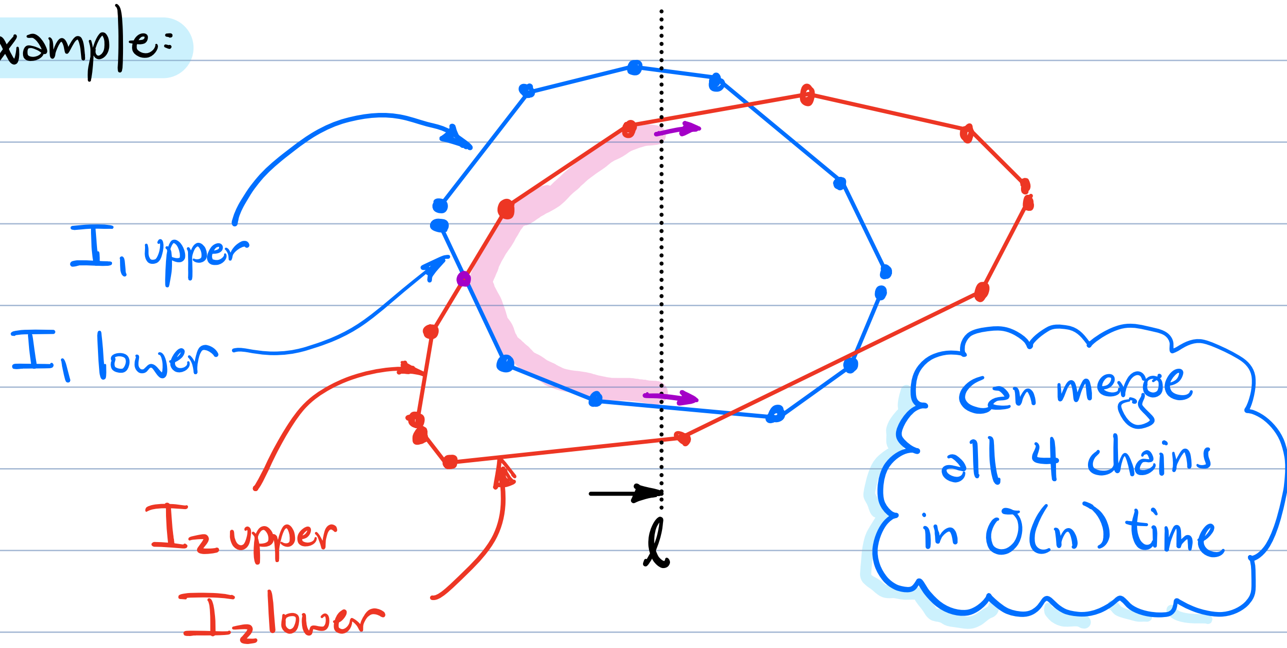
```
Intersect (H = {h_1, ..., h_n})
  if (|H| = 1) return h_1          // single halfplane
  else
    partition H as H_1 + H_2, each size ~ n/2
    I_1 ← Intersect (H_1)          // solve recursively
    I_2 ← Intersect (H_2)
    return merge (I_1, I_2)        // combine
```

How?

We can merge via plane sweep

- Each convex polygon viewed as upper + lower chains
- Merge these 4 chains, keeping
  - lower of the two upper chains
  - upper of the two lower chains

Example:



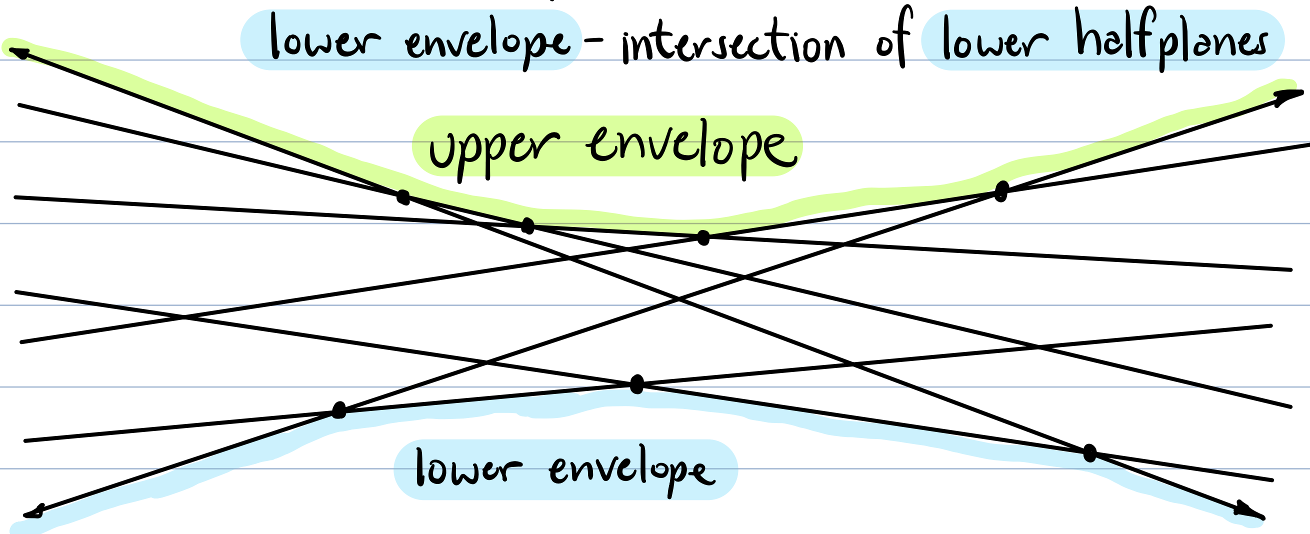
Total time: $$T(n) = \underbrace{2T(n/2)}_{\text{2 recursive calls on inputs of size } n/2} + \underbrace{n}_{\text{time to merge}}$$

$$= O(n \log n) \quad \text{[classical result]}$$

Envelopes:

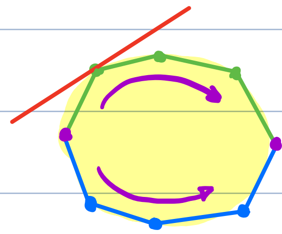
Given a set of lines (none vertical) define:

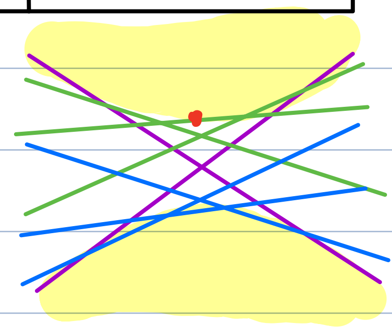
- **upper envelope** – intersection of upper halfplanes
- **lower envelope** – intersection of lower halfplanes



- This structure is reminiscent of the upper & lower hulls in Graham scan.

	Convex hulls	Envelopes
Input	Points	Lines
Order	By x-coord	By slope
Support	Line	Point



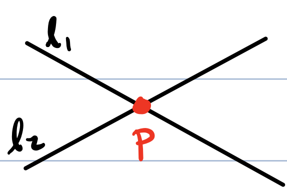


Can we formalize these similarities?

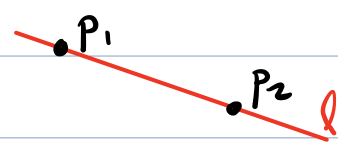
Point-Line Duality: Lines & points share similarities.

2 degrees of freedom: $\ell: y = ax + b$ $\quad p = (a, b)$

Incidence:

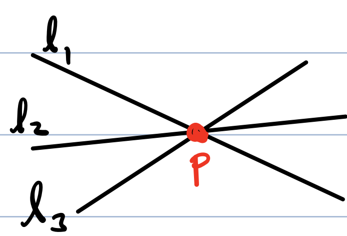


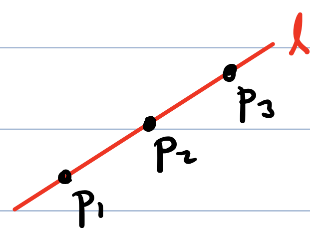
$p = \ell_1 \wedge \ell_2$



$\ell = p_1 \vee p_2$

Degeneracy:





Duality: Given point $p = (a,b)$ $\quad a, b \in \mathbb{R}$

$+$ line $\ell: y = cx - d$ $\quad c, d \in \mathbb{R}$

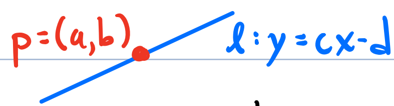
Define duals: p^* is the line: $y = ax - b$

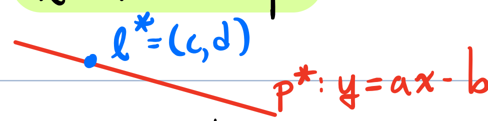
ℓ^* is the point: (c,d)

Properties:

Self-inverse: $p^{**} = p$ $\qquad \ell^{**} = \ell$

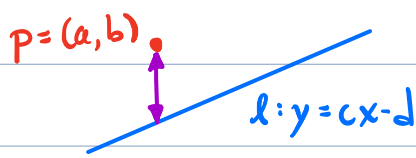
Incidence preserving: p lies on ℓ iff ℓ^* lies on p^*

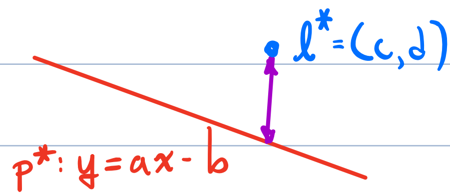




$b = c \cdot a - d \iff d = a \cdot c - b$

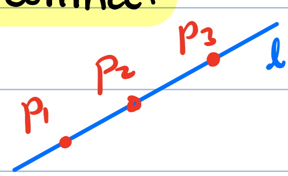
Preserves/Reverses vertical distances: p is at dist δ above ℓ iff ℓ^* is at dist δ above p^*

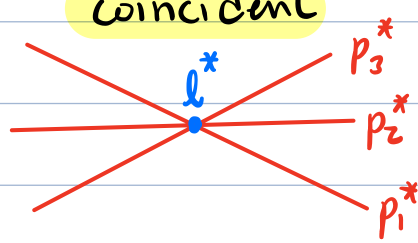




$b = c \cdot a - d + \delta \iff d = a \cdot c - b + \delta$

Degeneracy: p_1, p_2, p_3 collinear iff p_1^*, p_2^*, p_3^* coincident



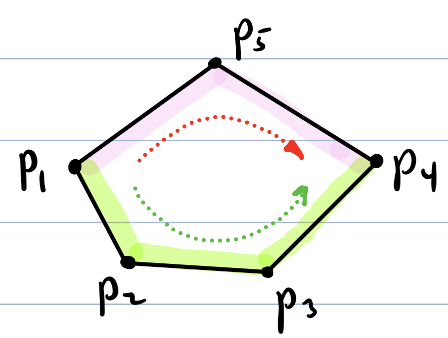


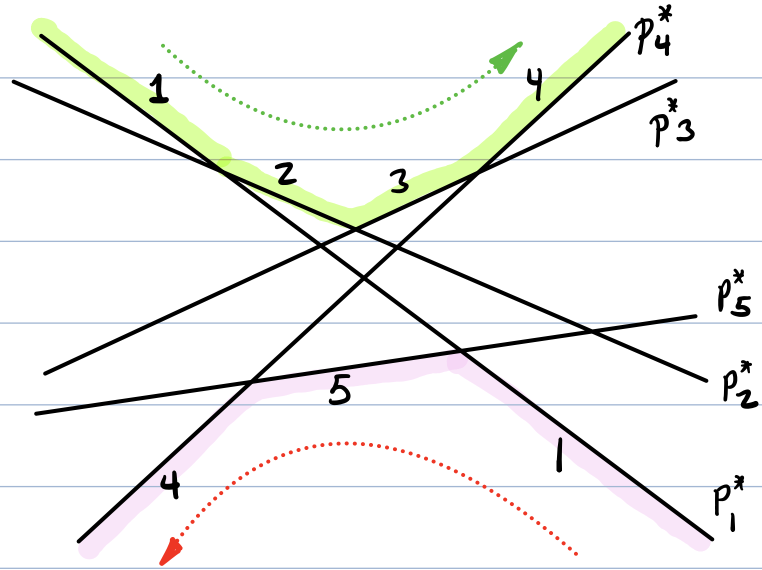
Hulls + Envelopes:

Lemma:

Given a point set $P = \{p_1, \dots, p_n\}$, the vertices of the upper/lower hull of $\mathrm{conv}(P)$, sorted left to right are in 1-1 correspondence with edges of lower/upper envelope of $P^* = \{p_1^*, \dots, p_n^*\}$, sorted in increasing slope order.

Example:

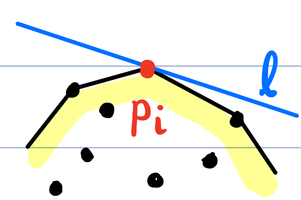




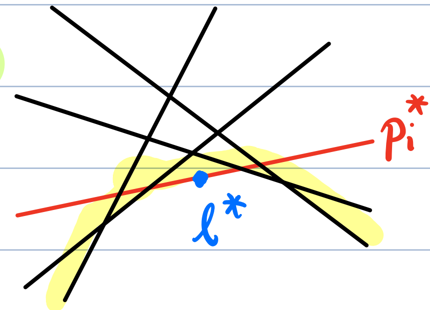
Proof:

p_i is on upper hull of $\mathrm{conv}(P)$

$\Leftrightarrow$ $\exists$ supporting line ℓ through p_i s.t all pts of P are on or below ℓ



$\Leftrightarrow$ $\exists$ point ℓ^* through p_i^* s.t. all lines of P^* are on or above ℓ^*



$\Leftrightarrow$ p^* contributes an edge to lower envelope of P^*

Since x-coords map to line slopes in dual, the left-to-right order on hull $\equiv$ slope order in envelope. $\square$

Graham's scan for computing envelopes?

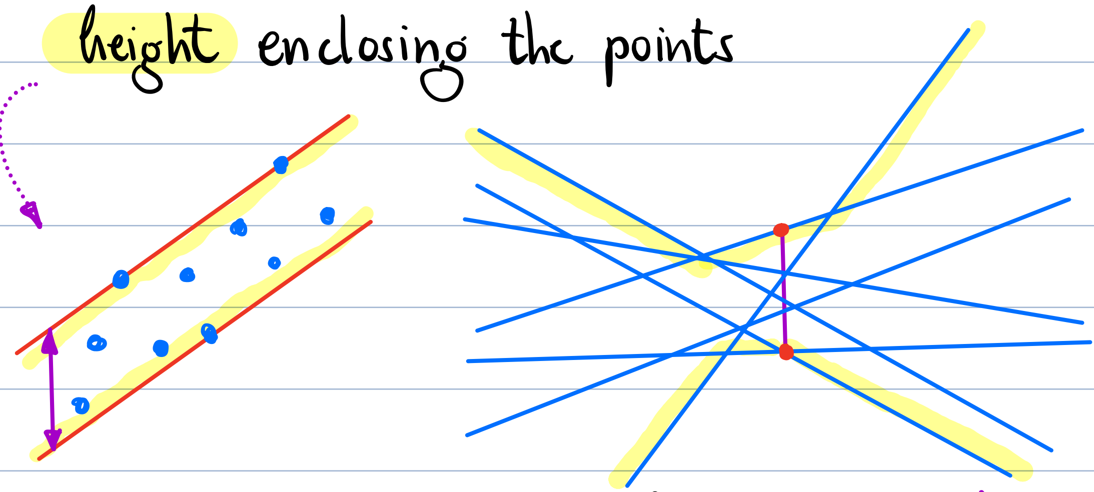
- This suggests that we can modify Graham's scan to compute envelopes
- In fact, no modification is needed!
- We just interpret inputs/outputs differently

	Hulls	Envelopes
Input (a_i, b_i)	Point (a_i, b_i)	Line: $y = a_i x - b_i$
Output	Indices of upper/lower hull vertices	Indices of lines on lower/upper env.

Applications:

- Duality allows us to couch problems in an alternative form, which may be easier or more intuitive

Narrowest slab: Given a set of n points, find slab (parallel lines) of minimum height enclosing the points



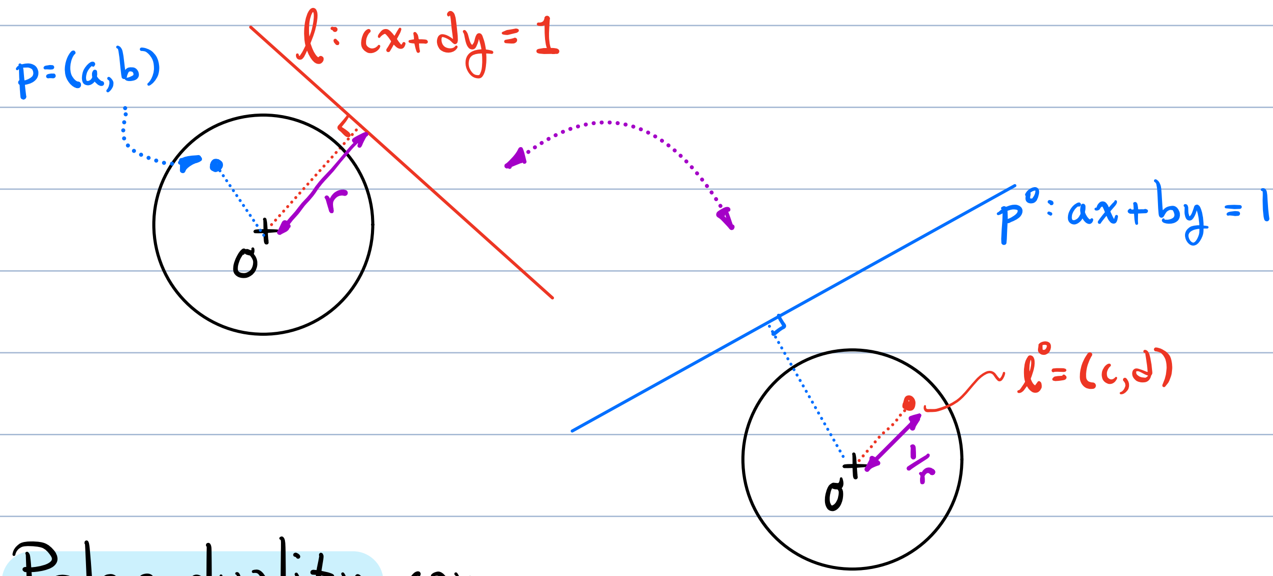
Thinnest cut: Given a set of n lines, find shortest vertical segment that cuts them all.

Polar Dual: Different line representations yield duality mappings with different properties.

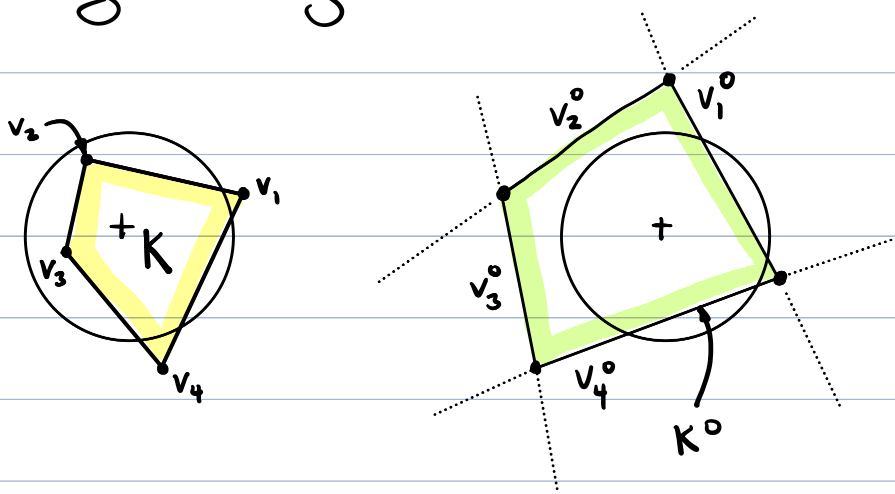
Polar dual:

$$p = (a, b) \longleftrightarrow p^{\circ} : ax + by = 1$$

$$\ell : cx + dy = 1 \longleftrightarrow \ell^{\circ} : (c, d)$$



Polar duality can be applied to vertices of a convex polygon (containing the origin)



The polar body K° has many interesting relationships with original K.

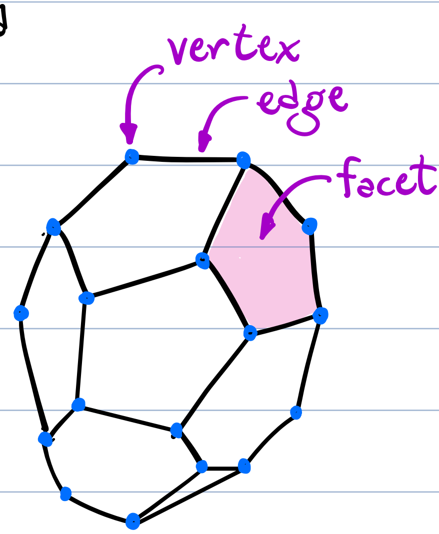
- Dual structure (in any dimension)

vertices	$\xleftrightarrow{1\text{-}1}$	faces
edges	$\xleftrightarrow{1\text{-}1}$	edges
faces	$\xleftrightarrow{1\text{-}1}$	vertices

- Mahler Volume: $\mathrm{vol}(K) \cdot \mathrm{vol}(K^\circ) = \mathcal{O}(1)$

Polytopes in Higher Dimensions

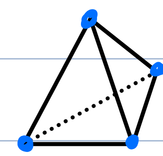
- Consider a convex polytope K in $\mathbb{R}^d$
- For $0 \le i \le d$, let f_i denote the number of i-dimensional faces:



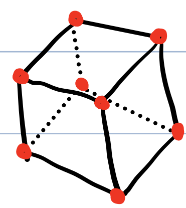
$i = 0$	– vertices
1	– edges
2	⋮
⋮	
$d-1$	– facets
d	– K itself

Euler's Formula: In $\mathbb{R}^3$: $f_0 - f_1 + f_2 = 2$

$4 - 6 + 4 = 2$



$8 - 12 + 6 = 2$



In higher dimensions, the numbers grow exponentially with dimension.

Upper-Bound Theorem (McMullen, 1970)

A convex polytope in $\mathbb{R}^d$ with n vertices can have $O(n^{\lfloor d/2 \rfloor})$ facets (+ vice versa) + this bound is tight (cyclic polytopes)

Summary:

- Halfplane intersection (in $\mathbb{R}^2$) - $O(n \log n)$
- Upper + lower envelopes of lines
- Point-line duality + properties
- Upper/Lower hulls $\Leftrightarrow$ Lower/Upper Env.
- Polar duality
- Euler's Formula + McMullen's Thm