Name (PRINTED): ____________________

Student ID #: ____________________

Section # (or TA's: name and time) ____________________

CMSC 250 Exam #1 Answers Thursday, Mar. 14, 2001

1. [20 pnts.]

Use a truth table to determine if the following argument is valid. Use 1 for "true" and 0 for "false" to create the complete truth table. If it is not valid, indicate which row/column indicate that it is not valid; if it is valid, mark all rows/columns which prove it is valid.

P1	$p \leftrightarrow w$
P2	$p \vee \sim q$
P3	$\sim p \rightarrow w$
	Therefore $\sim p \wedge q$

p	q	w	$p \leftrightarrow w$	$p \vee \sim q$	$\sim p \rightarrow w$	$\sim p \wedge q$	
1	1	1	1	1	1	0	Critical
1	1	0	0	1	1	0	
1	0	1	1	1	1	0	Critical
1	0	0	0	1	1	0	
0	1	1	0	0	1	1	
0	1	0	1	0	0	1	
0	0	1	0	1	1	0	
0	0	0	1	1	0	0	

No, these statements do not represent a valid argument.

2. [20 pnts.]

Use only those rules given on the "cheatsheet" to prove that the following is a valid argument. It is a Valid Argument - you only need to prove that it is.

P1	$(s \wedge q) \leftrightarrow w$
P2	$\sim s \rightarrow r$
P3	$\sim p \vee q$
P4	$\sim (\sim p \vee r)$
	Therefore $x \rightarrow w$

line	Statement	Reason	Line #s
1	$p \wedge \sim r$	De Morgan's	P4
2	p	$\wedge$ Simplification	1
3	$\sim r$	$\wedge$ Simplification	1
4	q	$\vee$ Syllogism	2, P3
5	$\sim\sim s$	Modus Tollens	3, P2
6	s	Double Negation	5
7	$s \wedge q$	$\wedge$ Addition	4,6
8	w	Modus Ponens (Maybe need $\leftrightarrow$ stuff?)	7, P1
9	$\sim x \vee w$	$\vee$ Addition	8
10	$x \rightarrow w$	Def. $\rightarrow$	9
11			
12			
13			
14			

3. [20 pnts.]For each of the following give the requested evidence for the relationship stated:

a. Disprove the following statement by giving a counter example (it is false):

$$\forall a, b \in \mathbf{Z}^{\mathbf{even}}, \text{ if } a \equiv b \bmod 3, \text{ then } a^2 \equiv b^2 \bmod 9.$$

Counterexample: $a = 2$, $b = 8$. $2 \equiv 8 \bmod 3$, $2^2 = 4$, and $8^2 = 64$, and $64 \equiv 1 \bmod 9$, while $4 \equiv 4 \bmod 9$.

b. Give a complete proof of the following statement (it is true):

$$\forall a, b \in \mathbf{Z}^{\mathbf{even}}, \text{ if } a \equiv b \bmod 4, \text{ then } a^2 \equiv b^2 \bmod 16.$$

Here are a few basic ways to do this. (There may be others)

i. You could Let $a - b = 4k$, then since a and b are even, $a = 2m, b = 2n$, and get that $n - m$ is even and use the result in class that says if $n - m$ is even, then $n + m$ is even, to show that $a + b = 4l$, so $a^2 - b^2 = (4k)(4l)$.

ii. You could break it into cases so either $a = 4k$ and $b = 4l$, or $a = 4k + 2$ and $b = 4l + 2$.

iii. You could say that $a = 4k + b$, then square both sides to get $a^2 = 16k^2 + 8kb + b^2$, and then since $b = 2l$ is even, we have that $a^2 = 16k^2 + 16kl + b^2$.

I will do the first option.

Proof: Since $a \equiv b \bmod 4$, $\exists k \in \mathbf{Z}$ such that $a - b = 4k$. Because a and b are even, $\exists m, n \in \mathbf{Z}$ such that $a = 2m$ and $b = 2n$.

Hence, $a - b = 2m - 2n = 4k$. Then, $m - n = 2k$.

From a result in class, since $n - m$ is even, and n and m are integers, $n + m$ is even. Hence $\exists l \in \mathbf{Z}$ such that $m + n = 2l$.

$a + b = 2m + 2n = 2(m + n) = 2(2l)$.

$a^2 - b^2 = (a - b)(a + b) = (4k)(4l) = 16kl$.

Since, $k, l \in \mathbf{Z}$, $kl \in \mathbf{Z}$, which means that a^2 and b^2 differ by an exact multiple of 16.

Hence $a^2 \equiv b^2 \bmod 16$.

4. [10 pnts.]

Fill in all of the blanks to show the subtraction of numbers (through binary addition using 8-bit 2's-compliment). The purpose of this is to show in a very step by step fashion that the subtraction of 8-bit 2's-compliment represented binary numbers does work.

Solve the problem of 7 - 18 = ??.

1	The 8-bit binary representation of the decimal value 18 =	00010010
2	The 8-bit 2's complement representation of the decimal value -18 =	11101110
3	The 8-bit binary representation of the decimal value 7 =	00000111
4	Use Binary addition to add lines 2 and 3. 2's complement representation of result =	11110101
5	The 8-bit binary representation of the decimal value 11 =	00001011
6	The 8-bit 2's complement representation of the decimal value -11 =	11110101
7	The values in lines 4 and 6 are (circle 1)	SAME

5. [20 pnts.] For each of the following English sentences, translate the meaning into formal notation using the logic symbols ($\exists$, $\forall$, $\wedge$, $\vee$, $\sim$, and $\rightarrow$). In addition to these, you may also use mathematical, grouping and set notations symbols as needed. On the next line write the negation of the original statement using formal notation.

There are at least 2 fields that are green and in the sun. Domains: F = {all fields} Predicates: S(x) = "x is in the sun" and G(x) = "x is green"
statement: $\exists x, y \in F,\ (x \neq y \wedge G(x) \wedge G(y) \wedge S(x) \wedge S(y))$
negation: $\forall x, y \in F\ (x \equiv y \vee \sim G(x) \vee \sim G(y) \vee \sim S(x) \vee \sim S(y))$

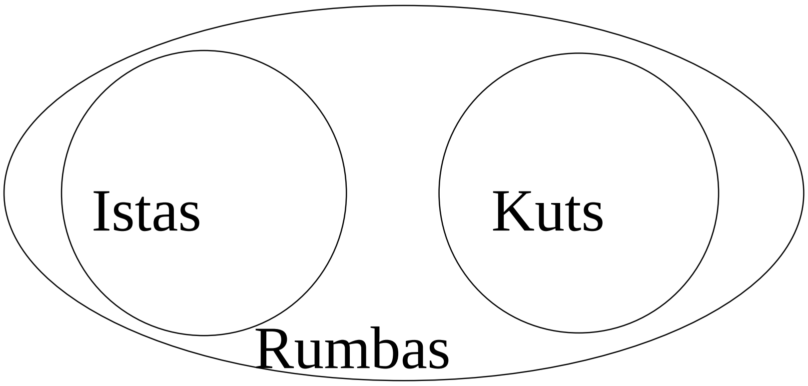
Thre is at least 1 real number between any two consecutive integers. Domain: R = {all reals} and Z = {all integers} Predicate: L(x,y)= "x is less than y"
statement: $\forall n \in \mathbf{Z} \exists \mathbf{x} \in \mathbf{R},\ \mathbf{L}(\mathbf{x}, \mathbf{n+1}) \wedge \mathbf{L}(\mathbf{n}, \mathbf{x})$
negation: $\exists n \in \mathbf{Z} \forall \mathbf{x} \in \mathbf{R},\ \sim \mathbf{L}(\mathbf{x}, \mathbf{n+1}) \vee \sim \mathbf{L}(\mathbf{n}, \mathbf{x})$

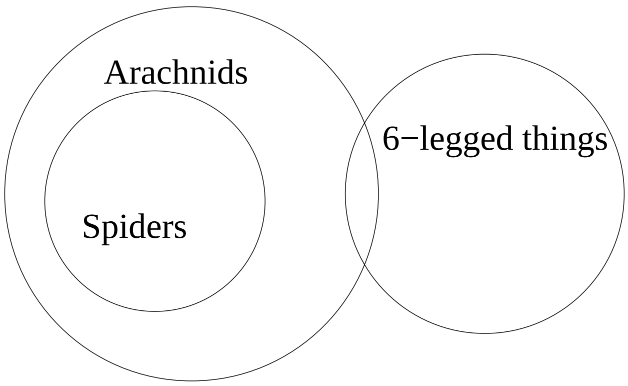
There is at most 1 even prime. Domains: Z = {all integers} Predicate: E(a) = "a is even" and P(a) = "a is prime"
statement: $\forall m, n \in \mathbf{Z},\ (\mathbf{P}(\mathbf{m}) \wedge \mathbf{P}(\mathbf{n}) \wedge \mathbf{E}(\mathbf{m}) \wedge \mathbf{E}(\mathbf{n})) \rightarrow \mathbf{m} = \mathbf{n}.$
negation: $\exists m, n \in \mathbf{Z},\ (\mathbf{P}(\mathbf{m}) \wedge \mathbf{P}(\mathbf{n}) \wedge \mathbf{E}(\mathbf{m}) \wedge \mathbf{E}(\mathbf{n}) \wedge \mathbf{m} \neq \mathbf{n}).$

Every positve integer has a real value that is its square root. Domains: Z = {all integers} and R = {all reals} Predicate: P(a) = "a is positive" and S(a,b) = "$a = \sqrt{b}$"
statement: $\forall n \in \mathbf{Z} \exists \mathbf{r} \in \mathbf{R},\ \mathbf{P}(\mathbf{n}) \rightarrow \mathbf{S}(\mathbf{n}, \mathbf{r})$
negation: $\exists n \in \mathbf{Z} \forall \mathbf{r} \in \mathbf{R},\ (\mathbf{P}(\mathbf{n}) \wedge \sim \mathbf{S}(\mathbf{n}, \mathbf{r}))$

6. [10 pnts.]

Use an Euler diagram to determine if each of the following represents a valid argument form. Make sure to label the parts of the diagram. If it is invalid, you must draw a diagram that is not supportive of the conclusion. If it is a valid argument, draw a diagram that does support the conclusion (since you can't draw one that doesn't).

All istas are rumbas. All kuts are rumbas. ——— therefore: Some istas are kuts.
The argument is Invalid.


No spiders have 6 legs. All spiders are arachnids. ——— therefore: No 6 legged animals are arachnids.
The argument is Invalid.


7. [20 pnts.] Use only the rules provided on the "cheatsheet" to prove the following. It is a Valid Argument - you only need to prove that it is. You may also assume the Domain is not empty (it has at least one member).

P1	$\forall x \in D \;\; P(x) \rightarrow R(x)$
P2	$\forall y \in D \;\; Q(y) \vee \sim R(y)$
P3	$\exists x \in D \;\; \sim Q(x) \wedge M(x)$
	−−−−−−−−−−−−−−−−
	therefore $\exists z \in D \;\; M(z) \wedge \sim P(z)$

line	Statement	Reason	Line #s
1	$\sim Q(a) \wedge M(a)$	$\exists$ Instantiation	P3
2	$\sim Q(a)$	$\wedge$ Simplification	1
3	$M(a)$	$\wedge$ Simplification	1
4	$Q(a) \vee \sim R(a)$	$\forall$ Instantiation	P2
5	$\sim R(a)$	$\vee$ Syllogism	2,4
6	$\sim P(a)$	$\forall$ Modus Tollens	5,P1
7	$M(a) \wedge \sim P(a)$	$\wedge$ Addition	3,6
8	$\exists z \in D \;\; M(z) \wedge \sim P(z)$	$\exists$ Generalization	7
9			
10			
11			
12			
13			
14			

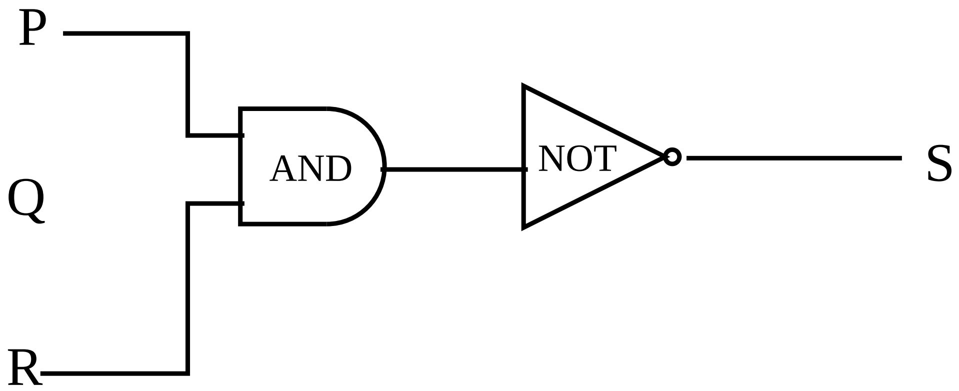
8. [20 pnts.]

Construct a circuit using the fewest number of 2-input AND, 2-input OR, and NOT gates that has an output S given by the following truth table:

P	Q	R	S
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

Give the reduced logical statement represented by the circuit: ____________________$\sim (P \wedge R)$________________

Note that we will also accept other answers that are close: such as $\sim P \vee \sim (P \wedge \sim R)$, $\sim P \vee (P \wedge \sim R)$, or $\sim P \vee \sim R$.



9. [20 pnts.]

For all integers n, if $n \equiv 3 \bmod 7$, then $\frac{n^2+5}{7} = \lfloor \frac{n^2+6}{7} \rfloor$

Proof:

Since $n \equiv 3 \bmod 7$, $\exists k \in \mathbf{Z}$ such that $n = 7k + 3$.

Then, $n^2 = (7k+3)^2 = 49k^2 + 42k + 9$.

Then, $n^2 + 5 = 49k^2 + 42k + 9 + 5 = 49k^2 + 42k + 14$,

and finally $\frac{n^2+5}{7} = 7k^2 + 6k + 2$. Refer to this value as **.

From this, we know that $\frac{n^2+6}{7} = \frac{n^2+5+1}{7} = \frac{n^2+5}{7} + \frac{1}{7} = 7k^2 + 6k + 2 + \frac{1}{7}$.

This means that $\lfloor \frac{n^2+6}{7} \rfloor = \lfloor 7k^2 + 6k + 2 + \frac{1}{7} \rfloor$.

Since $k \in \mathbf{Z}$, $7k^2 + 6k + 2 \in \mathbf{Z}$, which means that $\lfloor 7k^2 + 6k + 2 + \frac{1}{7} \rfloor = 7k^2 + 6k + 2 + \lfloor \frac{1}{7} \rfloor + 7k^2 + 6k + 2$, which is also equal to ** from the previous calculation.

Hence, $\frac{n^2+5}{7} =$**$= \lfloor \frac{n^2+6}{7} \rfloor$.

10. [20 pnts.]

Give a complete proof to show that the following statement is true.

$$\forall a, b, c \in Z^+ \ \forall p \in Z^{prime} \ \ [(a^2 = b^3 = c) \wedge (p|c)] \rightarrow (p^6|c)$$

Proof:

Because $p|c$, and $c = a^2$, $p|a^2$, which means that $p|a$, since p is prime.

Because $p|c$ and $c = b^3$, $p|b^3$, which means that $p|b$, since p is prime.

By the Unique Factorization Theorem, we can factor a b, and c into a products of primes:

$a = p^{e_a}a_1$, where $p \not| a_1$, and $e_a > 0$ because $p|a$.

$b = p^{e_b}b_1$, where $p \not| b_1$, and $e_b > 0$ because $p|b$.

$c = p^{e_c}c_1$, where $p \not| c_1$, and $e_c > 0$ because $p|c$.

Hence $c = a^2 = (p^{e_a}a_1)^2 = p^{2e_a}a_1^2$,

and $c = b^3 = (p^{e_b}b_1)^3 = p^{3e_b}b_1^3$.

Because we have unique factorization into primes,

$e_c = 2e_a$ and $e_c = 3e_b$.

Hence, $2e_a = 3e_b$.

Thus, $3|e_a$ which means that $\exists k \in \mathbf{Z}$ such that $e_a = 3k$.

Hence, $e_c = 2(e_a) = 2(3k) = 6k$. Since $p|c$, $6k > 0$, $k > 0$, which means that $6k \geq 6$.

This means that we can write c as: $c = p^6(p^{6-e_c}c_1)$, and observe that since $6 - e_c \geq 0$, $p^{6-e_c}c_1$ is an integer.

Hence, $c = p^6n$ for some $n \in \mathbf{Z}$, which means that $p^6|c$.