CMSC 858T, Randomized Algorithms, Spring 2003: Final Exam

Instructions: There are **three** questions. If you think you have some ideas that deserve partial credit, please *itemize them concisely.* Also, if you are unable to solve a problem but can do so by making some assumptions, state these assumptions clearly and proceed. Good luck!

1. We are given some positive constant $c < 1/2$.

(a) Show that there are positive constants a and b (which may be functions of c) such that for all large enough n, there is a graph $G = (V, E)$ on n vertices such that the following two properties hold. (i) The number of edges in G is at most an, and (ii) for *any* two *disjoint* sets of vertices A and B such that $|A| = |B| = cn$, there are at least bn edges which have one end-point in A and one end-point in B. (**Hint:** Consider the random graph model $G(n, p)$ where p is of the form $\Theta(1/n)$.) **(8 points)**

(b). Next, show that there is a graph G satisfying all the requirements of part (a), such that G has the additional property that it has no triangles. **(5 points)**

2. Let $u = (u_1, u_2, \ldots, u_d)$, $v = (v_1, v_2, \ldots, v_d)$, and $w = (w_1, w_2, \ldots, w_d)$ be three points in d-dimensional space, `where all the` u_i, v_i `and` w_i `lie in the set` $\{0, 1\}$.

(a) Call the ordered triple (u, v, w) *bad* if the dot product $(v - u) \cdot (w - u)$ equals zero. Using the fact that all the u_i, v_i and w_i lie in $\{0, 1\}$, give a simple necessary and sufficient condition for (u, v, w) to be bad. (**Hint:** Can $(v - u) \cdot (w - u)$ ever be negative?) **(3 points)**

(b) For as large a function $f(d)$ as you can come up with, show the following: there is a set S of $f(d)$ points in d-dimensional space where each point has all its co-ordinates lying in the set $\{0, 1\}$, such that for any ordered triple of three distinct points (u, v, w) chosen from S, the triple (u, v, w) is **not** bad. Justify your answer. **(8 points)**

3. We consider the Maximum Satisfiability problem (MAX-SAT) here. Given a Boolean formula $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ in conjunctive normal form with n Boolean variables $x_1, x_2, \ldots, x_n$ and m clauses $C_1, C_2, \ldots, C_m$, the objective is to find a Boolean assignment to the variables that maximizes the total number of satisfied clauses. Note that each C_i is an "or" of some of the literals: e.g., C_i could be $x_4 \vee \overline{x_7} \vee x_9 \vee x_{14}$. Let $|C_i|$ denote the number of literals in C_i; let $P(i)$ be the set of indices of the unnegated variables appearing in C_i, and $N(i)$ be the set of indices of the negated variables in C_i. (For instance, if C_i is "$x_4 \vee \overline{x_7} \vee x_9 \vee x_{14}$", then $|C_i| = 4$, $P(i) = \{4, 9, 14\}$, and $N(i) = \{7\}$.) Convince yourself that the following is a valid integer programming formulation for MAX-SAT. We have variables $y_1, y_2, \ldots, y_n$ and $z_1, z_2, \ldots, z_m$, all constrained to lie in $\{0, 1\}$. For each z_i, we have the constraint "$z_i \leq (\sum_{j \in P(i)} y_j) + (\sum_{j \in N(i)} (1 - y_j))$". Subject to these constraints, the objective is to maximize $\sum_i z_i$.

Consider the LP relaxation of this formulation obtained by relaxing each y_j and z_i to be a real lying in $[0, 1]$. Let $\{y_j^*, z_i^*\}$ be the values of the variables in an optimal solution to the LP. Our randomized rounding process will be, independently for each j, to set $x_j := 1$ (i.e., make variable x_j True) with probability y_j^* and $x_j := 0$ (make variable x_j False) with the remaining probability of $1 - y_j^*$.

(a). For any clause C_i, show that $\Pr[C_i \text{ satisfied}] \geq 1 - (1 - z_i^*/|C_i|)^{|C_i|}$. **(5 points)**

(b). Let Z be the random variable denoting the number of satisfied clauses, and let v^* denote the optimal objective function value of the LP relaxation. Assuming the result of part (a) to be true, prove that $\mathbf{E}[Z] \geq c \cdot v^*$ for as large a constant c as you can. **(6 points)**

– End of exam –