CMSC 858T: Randomized Algorithms, Spring 2003
Mid-Term Exam

Instructions: There are three questions. If you think you have some ideas that deserve partial credit, please itemize them concisely. Also, if you are unable to solve a problem but can do so by making some assumptions, please clearly state these assumptions and then proceed. Good luck!

1. (5 points) In this problem, we will view a permutation \( \pi \) of \( \{1, 2, \ldots, n\} \) as a sequence of \( n \) distinct numbers \( \pi_1, \pi_2, \ldots, \pi_n \), each of which lies in \( \{1, 2, \ldots, n\} \). You are given positive integers \( n \) and \( k \leq n \). Show that there is a collection \( \mathcal{F} \) of permutations of \( \{1, 2, \ldots, n\} \), with the following property:

   for any ordered sequence \((i_1, i_2, \ldots, i_k)\) of \( k \) distinct elements of the set \( \{1, 2, \ldots, n\} \), there is a permutation \( \pi \in \mathcal{F} \) such that \( \pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_k} \).

Important Note: The set \( \mathcal{F} \) in your solution should have a cardinality of the form \( f(k) \cdot \log n \), for some function \( f \).

2. We will consider tail bounds here.

   (a) (2 points) For any two random variables \( Y \) and \( Z \) such that \( Y, Z \in \[0, 1\] \), show that \( \text{coVar}[Y, Z] \leq \mathbb{E}[YZ] \leq (\mathbb{E}[Y] + \mathbb{E}[Z])/2 \). (Here, \( \text{coVar}[Y, Z] \) denotes the covariance of \( Y \) and \( Z \).)

   (b) (5 points) Use part (a) to show that there is a constant \( c > 0 \) such that the following holds.

   Suppose \( X_1, X_2, \ldots, X_n \) are random variables, each lying in \([0, 1]\), such that each \( X_i \) is dependent on at most \( d \) of the other \( X_j \). (For instance, if \( d = 3 \), then \( X_1 \) may depend on \( X_2, X_4, X_7 \) and be independent of the others, \( X_2 \) may depend on \( X_1, X_8 \) and be independent of the others, etc.) Let \( X = \sum_i X_i \) with \( \mu = \mathbb{E}[X] \). Then, for any \( \delta \in (0, 1) \),

   \[
   \Pr[X \leq \mu(1 - \delta)] \leq \frac{c \cdot d}{\mu \delta^2}.
   \]

3. We are given an undirected graph \( G = (V,E) \) with a non-negative weight \( w_i \) given for each vertex \( i \). We want to find a subset \( S \) of \( V \) with minimum total weight, in such a way that every edge of \( G \) has at least one end-point in \( S \).

   (a) (3 points) Formulate this problem as an ILP, and give its LP relaxation.

   (b) (5 points) Give a simple deterministic rounding scheme for this LP, to get a 2–approximation algorithm; justify why the approximation ratio of your algorithm is at most 2.