CMSC 858T: Randomized Algorithms

Spring 2003

Handout 7: Feng Guo's proof of a version of the FKG inequality

An important special case of the FKG inequality is that the intuitively apparent correlation among increasing/decreasing events in a product probability space, is rigorously true. Shortly after I taught the special case (without proof) in class, a student in class, Feng Guo, came up with the following proof for this special case. I am especially happy to post this here, with some minor editing.

Feng's proof is as follows. Suppose we have two events A, B and event $C = A \wedge B$. A, B are both completely determined by n independent binary variables $x_1, x_2, \cdots, x_n$. We want to prove that if both A and B are increasing, then $\Pr(C) \geq \Pr(A) \times \Pr(B)$.

Using induction, suppose when A and B are determined by $n-1$ variables $x_1, x_2, \cdots, x_{n-1}$ the statement holds. (The base case $n = 1$ is easy.) Define the following: $a_0 = \Pr(A|x_n = 0)$; $b_0 = \Pr(B|x_n = 0)$; $c_0 = \Pr(A \wedge B|x_n = 0)$; $a_1 = \Pr(A|x_n = 1)$; $b_1 = \Pr(B|x_n = 1)$; $c_1 = \Pr(A \wedge B|x_n = 1)$. Since A, B are increasing, we have $a_1 \geq a_0$; $b_1 \geq b_0$. Because of the inductive hypothesis, $c_0 \geq a_0 b_0$ and $c_1 \geq a_1 b_1$.

Denote $\Pr(x_n = 1) = p$; thus, $\Pr(x_n = 0) = 1 - p$. Note that

$$\Pr(C) = c_1 p + c_0(1-p); \;\; \Pr(A) = a_1 p + a_0(1-p); \;\; \Pr(B) = b_1 p + b_0(1-p).$$

Our target becomes to prove:

$$c_1 p + c_0(1-p) \geq (a_1 p + a_0(1-p))(b_1 p + b_0(1-p)).$$

As stated before, $c_1 p + c_0(1-p) \geq a_1 b_1 p + a_0 b_0 (1-p)$; so, it would be enough to prove

$$a_1 b_1 p + a_0 b_0(1-p) \geq a_1 b_1 p^2 + a_0 b_0 (1-p)^2 + (a_0 b_1 + a_1 b_0) p(1-p).$$

This reduces to $p(1-p)(a_1 - a_0)(b_1 - b_1) \geq 0$, which is true.