

CMSC 858T: Randomized Algorithms

Spring 2003

Homework Assignment #2 (will be graded)

Due date: Beginning of class on March 18, 2003

Note: Please submit only a final, polished set of answers, and do not include unnecessary material. (For instance, please don't include your preliminary work/calculations/intuition that led to the final answer.) Partial credit will be given where appropriate: if you think you have some ideas that deserve partial credit, please *itemize them concisely*, so that the grading process can be accurate. You are welcome to collaborate with your fellow-students; you must still write up the solutions by yourself, and list your collaborators. The problem marked (*) may be more difficult than the others.

1. Recall the low-congestion routing problem studied in class, and let \mathcal{P}_i be the given set of candidate paths connecting s_i and t_i , for each i . Consider the simple deterministic rounding strategy \mathcal{D} that we considered briefly: for each (s_i, t_i) -pair, choose a path from \mathcal{P}_i which has the highest value given by the LP (breaking ties arbitrarily), and use this path to connect s_i and t_i . Suppose we have an input instance in which for each i , \mathcal{P}_i has at most a paths for some parameter a . Bound the approximation guarantee of \mathcal{D} on such input instances. (That is, find as small a ρ as you can, such that the congestion achieved by \mathcal{D} on such input instances, is at most ρ times the optimal congestion.)

2. Consider the following minimization problem. We are given an undirected graph $G = (V, E)$, and have to choose some vertices at which to place facilities (one facility per chosen vertex) such that the following holds: for each vertex v , some facility is placed at v or at at least one of the neighbors of v (as usual, the "or" here is an inclusive or, not an exclusive or). We want to place a minimum number of facilities such that this holds.

- Formulate this as an ILP, by introducing a 0-1 variable x_i for whether we place a facility at vertex i or not. Write down the LP relaxation of the ILP.
- Instead of rounding *down* as in edge-disjoint paths, we will now round *up* by some $\lambda \geq 1$, as follows. If $x_i^* \geq 1/\lambda$, round x_i deterministically to 1; else if $x_i^* < 1/\lambda$, round x_i to 1 independently with probability λx_i^* . Do a union bound over the constraints of the problem and apply Markov's inequality to the objective function (this is simpler than applying the Chernoff bound); what is the smallest approximation ratio that you can achieve by such an approach with probability at least 0.1, say?

3. Suppose we generate a random graph G from the random graph model $G(2t, 1/2)$, for some integer t . Show that the probability that all vertices in G have degree at most $t - 1$, is at least $1/4^t$.

4(*). We have an arbitrary connected graph G . Recall that a subgraph of G is any graph obtained by choosing some subset of the edges of G . Call a subgraph H of G *good* iff H is connected, and includes every vertex of G (i.e., H is a spanning connected subgraph of G). Suppose we generate a random subgraph H_1 of G by choosing each edge of G independently with probability $1/2$; let p_1 be the probability that H_1 is good. Now consider a different random experiment on G . Suppose we color the edges of G Red or Blue randomly as follows: each edge is independently colored, and the choice of Red or Blue for every edge is made with probability $1/2$ each. Let p_2 be the probability that

“the red edges form a good subgraph, **and** the blue edges form a good subgraph.”

Is it true that $p_2 \leq p_1^2$? Justify your answer.