CMSC 858T: Randomized Algorithms

Spring 2003

Homework Assignment #3 **(will be graded)**

Due date: Beginning of class on May 1, 2003

Note: Please submit only a final, polished set of answers, and do not include unnecessary material. (For instance, please don't include your preliminary work/calculations/intuition that led to the final answer.) Partial credit will be given where appropriate: if you think you have some ideas that deserve partial credit, please *itemize them concisely*, so that the grading process can be accurate. You are welcome to collaborate with your fellow-students; you must still write up the solutions by yourself, and list your collaborators. The problem marked (*) may be more difficult than the others.

1. Show that there is a constant $a > 0$ such that the following holds. We have an arbitrary graph $G = (V, E)$ with maximum degree Δ. Each vertex v also has a list of colors L_v; we want to color each vertex v with some color from L_v, so that we get a proper coloring (i.e., no two adjacent vertices get the same color). Prove that this is possible if the following holds: there is a non-negative value $b_{v,c}$ for all vertices v and all colors $c \in L_v$, such that:

- $\forall v, \ \sum_{c \in L_v} b_{v,c} = 1$; and
- $\forall (u, v) \in E, \ \sum_{c \in L_u \cap L_v} b_{u,c} \cdot b_{v,c} \leq a/\Delta$.

Try to get as large a value for the constant a as you can. (**Hint:** Do a suitable random coloring and apply the Local Lemma.)

2(*). Show that there is an integer constant $a > 0$ such that the following holds. We have an arbitrary graph $G = (V, E)$ with maximum degree Δ. Show that we can give a color from $\{1, 2, \ldots, a\Delta\}$ to each *edge* of G, so that the following hold:

- no two edges that share an end-point get the same color;
- no *even-length* cycle has only two colors given to its set of edges.

(**Hint:** Do a random coloring of the edges. Associate one bad event $A_{e,f}$ with every pair of edges e and f that share an end-point, and one bad event B_C with every even cycle C. Use the *asymmetric* version of the Local Lemma; thus, you need to come up with suitable values $x_{e,f}$ and x_C. Take $x_{e,f}$ a constant times $\Pr[A_{e,f}]$, and come up with a suitable choice for the values x_C.)

3. Recall the "canonical problem" presented in class, in the context of Janson's inequalities. Suppose we have an instance of such a problem, where the notation such as the ground set $\{1, 2, \ldots, n\}$, the sets $S_1, S_2, \ldots, S_m$, X, and $\mu = E[X]$, is the same as introduced in class. Suppose each S_i intersects at most d other S_j. Give an upper bound on $\Pr[X = 0]$ that is a function of μ and d alone.