CMSC 858T: Randomized Algorithms  
Spring 2003  
Project (Due date: Beginning of class on May 13, 2003)

Please note the following four points:

(i) Please submit only a final, polished set of answers, and do not include unnecessary material. Partial credit will be given where appropriate: if you think you have some ideas that deserve partial credit, please itemize them concisely, so that the grading process can be accurate.

(ii) There are five problems below. Some of these problems involve the Lovász Local Lemma and/or Janson’s inequalities, so in case you are unable to solve a problem right away, you may wish to wait until the above two topics are covered in class. Also, for problems 1, 2, and 3, you have the option of “buying a hint” at any time: if you buy the hint for a particular problem, your final score for that problem will be 75% of the score that you get for that problem.

(iii) You are welcome to collaborate with your fellow-students; you must still write up the solutions by yourself, and list your collaborators. If you decide to collaborate with other student(s) on a problem and if any one of you buys the hint for that problem, the above “75% rule” will apply to all of you for that problem.

(iv) If $X$ is a random real drawn uniformly at random from the interval $[-1, 1]$, then for any function $f$, 

$$E[f(X)] = (1/2) \cdot \int_{-1}^{+1} f(x) \, dx;$$

this is basically because the density function of $X$ is $1/2$.

Good luck!

1. Suppose we generate $n$ independent random variables $X_1, X_2, \ldots, X_n$, where each $X_i$ is a random real drawn uniformly at random from the interval $[-1, 1]$. Let $p_n$ be the probability that $\sum_{i=1}^{n} X_i^2 \leq 1$. Show that for any constant $\epsilon > 0$, $p_n \leq O(\epsilon^n)$. More precisely, show the following: for any constant $\epsilon > 0$, there exist constants $n_0$ and $c$ such that for all $n \geq n_0$, $p_n \leq c \cdot \epsilon^n$.

2. We are given an undirected graph $G = (V, E)$, where there are $n$ vertices and where the minimum degree of any vertex is some value $d$ which is at least $(\log n)^2$. We want to give a color from the set $\{1, 2, \ldots, t\}$ to each vertex, such that for every vertex $u$, every color $c \in \{1, 2, \ldots, t\}$ has been assigned to at least one neighbor of $u$. (That is, if we “stand at any node”, all colors are visible in the neighborhood.) We want to achieve this with as large a value of $t$ as we can; for instance, the case $t = 1$ is trivial. In what follows, $\ln x$ denotes the natural logarithm of $x$, as usual.

   Show that there is a function $f(n)$ with $\lim_{n \to \infty} f(n) = 0$ such that for any graph $G$, there is a coloring as desired above, for which $t \geq (1 - f(n)) \cdot \frac{d}{\ln n}$. Also, give a deterministic polynomial-time algorithm to compute the corresponding coloring.

3. Show that there is an integer constant $c > 0$ such that the following holds. For any $\Delta \geq 2$ and any undirected graph $G = (V, E)$ with maximum degree $\Delta$, there is a way to assign a color from the set $\{1, 2, \ldots, c\Delta^2\}$ to each edge of $G$, such that the following is true. Consider any simple path $P$ in $G$, of even length; then, the ordered sequence of colors that appear in the
first half edges of $P$, does not equal the ordered sequence of colors that appear in the second half edges of $P$. (In other words, suppose $P$ has length $2i$, and that the ordered sequence of colors as we traverse the edges of $P$, is $a_1, a_2, \ldots, a_{2i}$. Then, the string $(a_1, a_2, \ldots, a_i)$ is not identically equal to the string $(a_{i+1}, a_{i+2}, \ldots, a_{2i})$.)

4. We are given a sequence of elements $S = (a_1, a_2, \ldots, a_m)$, where each $a_i$ lies in the set $\{1, 2, \ldots, n\}$. Let $b_i$ be the number of times the integer $i \in \{1, 2, \ldots, n\}$ occurs in $S$, and let $v = \sum_{i=1}^{n} b_i^2$. Also, let $X_1, X_2, \ldots, X_n$ be $d$-wise independent random variables (for some integer $d$) such that each $X_j$ is $-1$ with probability $1/2$, and $1$ with probability $1/2$.

Consider the following algorithm:

```plaintext
sum := 0;
for $i := 1$ to $m$ do:
  $j := a_i$;
  sum := sum + $X_j$;
} 
Y := sum^2.
```

Find the smallest integer $d$ that you can, for which the following holds: for all $\alpha > 0$, $\Pr[|Y - v| \geq \alpha v] \leq O(1/\alpha^2)$.

5. We are given an undirected graph $G = (V, E)$, where $|V| = n$ and $|E| = m$; it is also true that the maximum degree of any vertex in $G$ is at most twice the minimum degree.

(a) Give a lower bound on the minimum degree and an upper bound on the maximum degree, in terms of $n$ and $m$.

(b) We are given some integer $t \leq m$. Suppose we choose each vertex independently with some probability $p$; let $X$ denote the number of edges, both of whose end-points are chosen. If we want $\mathbb{E}[X]$ to be $t$, what should the value of $p$ be?

(c) Consider the random process of (b), along with the value $p$ computed there. Give an upper bound on $\Pr[X \leq t/2]$; your upper bound should be of the form $e^{-a^b}$, for some positive constants $a$ and $b$. (Hint: The fact that the maximum degree of $G$ is not much more than the minimum degree, may help you.)