CMSC 858T: Randomized Algorithms

Spring 2003

Ungraded Homework Assignment #2, handed out March 4, 2003

Please note: We will have *ungraded* homework assignments such as this one, as well as ones that will be graded. I will post the solutions for all the assignments some time after they are handed out. You will get the most out of this course if you do your best to solve all the homework problems (whether they are graded or not) by yourself, or in collaboration with your classmates. The suggested deadline by which to finish this assignment is March 11th; since this assignment is ungraded, you don't need to turn it in – just compare your solutions with the solutions I give.

0. Read Handout 3.

1. Verify the bounds on $F^+(\mu, \delta)$ that we showed in class, for the cases $\delta \leq 1$ and $\delta > 1$.

2. Fix a language $L \in BPP$, and the corresponding randomized algorithm A; recall that whatever the input x is, A will be correct with probability at least 2/3, in deciding whether $x \in L$ or not. Now suppose we want to boost this probability of 2/3 to at least $1 - 2^{-r}$, for some $r \geq 2$. To do so, we run A on the input string x independently some odd number t of times, and take the majority outcome of A's outputs on these t runs. Give an upper bound on t that will suffice to achieve the "$1 - 2^{-r}$" guarantee.

3. Consider the same basic setting as in class, for the Chernoff bound: $X = \sum_{i=1}^{n} X_i$, where:

- all the X_i lie in $[0, 1]$ and are independent of each other;
- $\mathbf{E}[X_i] = \mu_i$ for each i, and $\mathbf{E}[X] = \mu = \sum_i \mu_i$.

Consider some $t > 0$. As with the Chernoff bound,

$$\mathbf{E}[e^{tX}] = \prod_{i=1}^{n} \mathbf{E}[e^{tX_i}] \leq \prod_{i=1}^{n} (1 + \mu_i(e^t - 1)).$$

Now, instead of upper-bounding the last product by

$$\prod_{i=1}^{n} e^{\mu_i(e^t - 1)}$$

as we did in class, upper-bound the product using the AM-GM inequality. Proceeding in this vein, what upper-tail bound do you get (i.e., given $\delta > 0$, what upper bound do you get on $\Pr[X \geq \mu(1 + \delta)]$?).

4. Give a high-probability upper bound on the maximum load on any bin in the "m balls, n bins" problem, for the entire range of $m = m(n)$. (The functions Δ^+ and Δ^- from Handout 3 could help you here.)

5. Suppose a random variable X takes values in the interval $[a, b]$, for some given real numbers a and b. Suppose we want to estimate the mean of X to within an additive error of ϵ by the sampling procedure discussed in class. How many samples are sufficient?