CMSC 858T: Randomized Algorithms

Spring 2003

Ungraded Homework Assignment #4, handed out May 1, 2003

The suggested deadline by which to finish this assignment is May 13th; since this assignment is ungraded, you don't need to turn it in – just compare your solutions with the solutions I give.

1. In this problem, we prove that the Janson inequality parameter Δ is at most $O(\log N)$ in the Garg-Konjevod-Ravi Group Steiner Tree algorithm for trees, as claimed in class. Recall from class that we have a tree T with root r and n nodes. There are k disjoint groups $S_1, S_2, \ldots, S_k$, all of which are sets of leaves; also, $\max_i |S_i| = N$. Specifically, we fix a group S_i, and want to show that Δ_i, the Janson inequality parameter for the set of leaves that correspond to S_i, is at most $\ln |S_i|$.

(a). Suppose $j, j' \in S_i$. We will say that $j \sim j'$ if and only if **(i)** $j \neq j'$ and **(ii)** the least common ancestor of j and j' in G is not the root r. If $j \sim j'$, let $lca(j, j')$ denote the least common ancestral *edge* of j and j' in T'. Show that

$$\Delta_i = \sum_{j,j' \in S_i:\ j \sim j', x_{lca(j,j')} > 0} \frac{x_{pe(j)} x_{pe(j')}}{x_{lca(j,j')}}.$$

(b). We will now prove the following key fact:

$$\text{If } x_{pe(j)} > 0, \text{ then } x_{pe(j)} \cdot \sum_{j' \in S_i:\ j \sim j'} \frac{x_{\mathrm{pe}(j')}}{x_{lca(j,j')}} \leq x_{\mathrm{pe}(j)} \ln(1/x_{\mathrm{pe}(j)}). \tag{1}$$

We now take some steps toward proving (1). Suppose $x_{\mathrm{pe}(j)} = z \in \langle 0, 1]$. We need some extra notation. Let $e_0, e_1, \ldots, e_t$ be the sequence of edges that we encounter as we walk up the tree starting from j; let $y_\ell = x_{e_\ell}$. Thus we have $z = y_0 \leq y_1 \leq y_2 \leq \cdots \leq y_t \leq 1$. Next, for $\ell = 0, 1, \ldots, \ell$, let $A_\ell = \sum_{j' \in (T(e_\ell) \cap S_i)} x_{\mathrm{pe}(j')}$. Then, it is not hard to see that the left-hand side in the statement of (1) equals

$$z \cdot \sum_{\ell=1}^{t} \frac{A_\ell - A_{\ell-1}}{y_\ell}. \tag{2}$$

The sum in (2) is clearly bounded by the maximum of the following optimization problem, whose variables are the y_ℓ and A_ℓ. (The optimization problem has a maximum since the domain is a polytope and since the objective function is continuous in the domain.)

$$OPT(z,t): \quad \text{maximize } \textstyle\sum_{\ell=1}^{t} \frac{A_\ell - A_{\ell-1}}{y_\ell} \text{ subject to}$$

$$\begin{aligned}
A_0 &= z; \\
y_0 &= z; \\
y_t &\leq 1; \\
A_\ell &\leq A_{\ell+1}, \quad \ell = 0, 1, \ldots, t-1; \\
y_\ell &\leq y_{\ell+1}, \quad \ell = 0, 1, \ldots, t-1; \\
A_\ell &\leq y_\ell, \quad \ell = 0, 1, \ldots, t. \qquad (3)
\end{aligned}$$

Constraint (3) holds since the following constraint (4) is a constraint in our LP relaxation:

$$\sum_{j \in (L(f) \cap S_i)} x_{\mathrm{pe}(j)} \leq x_f \quad \text{for every edge } f \text{ and every group } S_i. \tag{4}$$

Fix any feasible solution $\{y_\ell, A_\ell : \ell \geq 0\}$ to the above optimization problem.

- If v is the objective function value of this solution to the optimization problem, show that

$$v \leq 1 - z/y_1 + \sum_{\ell=1}^{t-1}(1 - y_\ell/y_{\ell+1}). \tag{5}$$

- Take any ℓ, $2 \leq \ell \leq t-1$. If we keep all variables but y_ℓ fixed, see when the r.h.s. of (5) is maximized. Start with this idea to show that

$$v \leq 1 - z/y_1 + \ln(1/y_1). \tag{6}$$

- Use (6) to show that $v \leq \ln(1/z)$. This will then prove (1).

(c). Show, using (1), that $\Delta_i \leq \ln |S_i|$.

2. We have a set V of n elements, and m distinct subsets $S_1, S_2, \ldots, S_m$ of V, each having cardinality t. Our goal is to choose a subset W of V with "many" elements, subject to the constraint that no S_i (for $i = 1, 2, \ldots, m$) be a subset of W.

Consider the following algorithm $\mathcal{A}$ for this problem. Let V be the set $\{1, 2, \ldots, n\}$. Independently for each $i \in V$, choose a number X_i uniformly at random from the set $\{1, 2, \ldots, n^3\}$. Now define a set W as follows: for each $i \in V$, $i \in W$ iff there is no set S_j such that: (i) $i \in S_j$, and (ii) for all $k \in S_j$, $X_i \geq X_k$.

(a). Show that $\mathcal{A}$ always produces a feasible solution to our problem.

(b). Suppose $i \in V$ lies in a_i of the sets $S_1, S_2, \ldots, S_m$. Show that the expected size of the set W produced by $\mathcal{A}$ is at least

$$(1/n^3) \cdot \sum_{i=1}^{n} \sum_{\ell=1}^{n^3} \left(1 - (\ell/n^3)^{t-1}\right)^{a_i}.$$

3. Suppose G is an undirected graph with maximum degree Δ. Each vertex u has a given color-list $L(u)$, such that $|L(u)| \geq \Delta + 1$; we want a valid "list coloring" (i.e., an assignment of one color from $L(u)$ to each u, such that adjacent vertices get different colors.) Convince yourself that G has a valid list-coloring in this case; in this problem, we explore *distributed* list-coloring in the synchronous round-by-round model. (Thus, in a given round, any vertex communicates only with its neighbors.) Consider the following distributed algorithm for list-coloring. A generic round proceeds using the following four steps:

(S1) Each yet-uncolored vertex first *wakes up* with probability 1/2; if it chooses to not wake up in this round, it does not nothing for the rest of this round. (Thus, the three steps below only refer to vertices that woke up this round.)

(S2) u chooses a *tentative color* at random from its current list $L(u)$.

(S3) Each vertex u that has some neighbor that chose the same tentative color as u, is called *unsuccessful*; all other (yet-uncolored) vertices that woke up this round, are called successful.

(S4) Each successful vertex v is permanently given its chosen tentative color c, and this color c is removed from $L(w)$ for all neighbors w of v such that $c \in L(w)$. The other vertices proceed to the next round.

Note that once a vertex gets a permanent color, it is never considered again.

(a) Show that step (S2) is well-defined: if u is yet-uncolored, show that $L(u) \neq \emptyset$.

(b) Show that (if and) when the algorithm terminates, we have a valid list-coloring.

(c) Suppose we have some t yet-uncolored vertices at the beginning of a round. Show that the expected number of yet-uncolored vertices at the end of that round is at most $3t/4$.