Algorithmic Complexity 2

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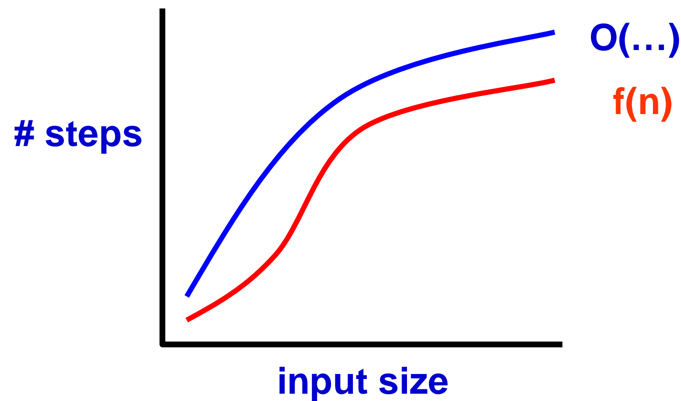
Overview

- Big-O notation
- Analysis cases
- Critical section

Big-O Notation

- Represents
  - Upper bound on number of steps in algorithm
  - Intrinsic efficiency of algorithm for large inputs



Formal Definition of Big-O

- Function f(n) is O(g(n)) if
  - For some positive constants M, N_0
  - $M \times g(N_0) \geq f(n)$, for all $n \geq N_0$
- Intuitively
  - For some coefficient M & all data sizes $\geq N_0$
    - $M \times g(n)$ is always greater than f(n)

Big-O Examples

- $5n + 1000 \Rightarrow O(n)$
  - Select $M = 6$, $N_0 = 1000$
  - For $n \geq 1000$
    - $6n \geq 5n+1000$ is always true
  - Example $\Rightarrow$ for $n = 1000$
    - $6000 \geq 5000 + 1000$

Big-O Examples

- $2n^2 + 10n + 1000 \Rightarrow O(n^2)$
  - Select $M = 4$, $N_0 = 100$
  - For $n \geq 100$
    - $4n^2 \geq 2n^2 + 10n + 1000$ is always true
  - Example $\Rightarrow$ for $n = 100$
    - $40000 \geq 20000 + 1000 + 1000$

Types of Case Analysis

- Can analyze different types (cases) of algorithm behavior
- Types of analysis
  - Best case
  - Worst case
  - Average case

Types of Case Analysis

- Best case
  - Smallest number of steps required
  - Not very useful
  - Example $\Rightarrow$ Find item in first place checked

Types of Case Analysis

- Worst case
  - Largest number of steps required
  - Useful for upper bound on worst performance
    - Real-time applications (e.g., multimedia)
    - Quality of service guarantee
  - Example $\Rightarrow$ Find item in last place checked

Types of Case Analysis

- Average case
  - Number of steps required for “typical” case
  - Most useful metric in practice
  - Different approaches
    1. Average case
    2. Expected case
    3. Amortized

Approaches to Average Case

1. Average case
   - Average over all possible inputs
   - Assumes uniform probability distribution
2. Expected case
   - Weighted average over all inputs
   - Weighted by likelihood of input
3. Amortized
   - Examine common sequence of operations
   - Average number of steps over sequence

Quicksort Example

- Quicksort
  - One of the fastest comparison sorts
  - Frequently used in practice
- Quicksort algorithm
  - Pick pivot value from list
  - Partition list into values smaller & bigger than pivot
  - Recursively sort both lists

Quicksort Example

- Quicksort properties
  - Average case = O(nlog(n))
  - Worst case = $O(n^2)$
    - Pivot ≈ smallest / largest value in list
    - Picking from front of nearly sorted list
- Can avoid worst-case behavior
  - Attempt to select random pivot value

Amortization Example

- Adding numbers to end of array of size k
  - If array is full, allocate new array
    - Allocation cost is O(size of new array)
  - Copy over contents of existing array
- Two approaches
  - Non-amortized
    - If array is full, allocate new array of size k+1
  - Amortized
    - If array is full, allocate new array of size 2k
  - Compare their allocation cost

Amortization Example

- Non-amortized approach
  - Allocation cost as table grows from 1..n

Size (k)	1	2	3	4	5	6	7	8
Cost	1	2	3	4	5	6	7	8

  - Total cost $\Rightarrow \frac{1}{2}(n+1)^2$
- Case analysis
  - Best case $\Rightarrow$ allocation cost = k
  - Worse case $\Rightarrow$ allocation cost = k
  - Average case $\Rightarrow$ allocation cost = k = n/2

Amortization Example

- Amortized approach
  - Allocation cost as table grows from 1..n

Size (k)	1	2	3	4	5	6	7	8
Cost	2	0	4	0	8	0	0	0

  - Total cost $\Rightarrow 2(n-1)$
- Case analysis
  - Best case $\Rightarrow$ allocation cost = 0
  - Worse case $\Rightarrow$ allocation cost = $2(k-1)$
  - Average case $\Rightarrow$ allocation cost = 2
- Worse case takes more steps, but faster overall

Analyzing Algorithms

- **Goal**
  - **Find asymptotic complexity of algorithm**
- **Approach**
  - **Ignore less frequently executed parts of algorithm**
  - **Find critical section of algorithm**
  - **Determine how many times critical section is executed as function of problem size**

Critical Section of Algorithm

- **Heart of algorithm**
- **Dominates overall execution time**
- **Characteristics**
  - **Operation central to functioning of program**
  - **Contained inside deeply nested loops**
  - **Executed as often as any other part of algorithm**
- **Sources**
  - **Loops**
  - **Recursion**

Critical Section Example 1

- Code (for input size n)
  1. A
  2. for (int i = 0; i < n; i++)
  3.   B
  4. C
- Code execution
  - A ⇒
  - B ⇒
  - C ⇒
- Time ⇒

Critical Section Example 1

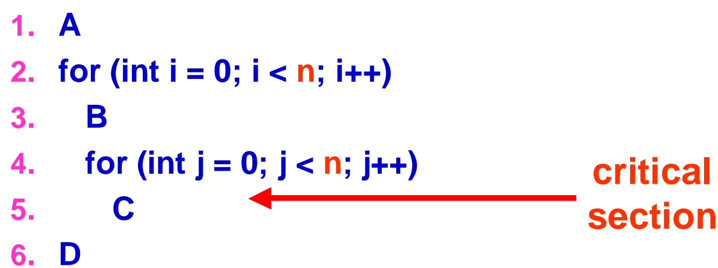
- Code (for input size n)
  1. A
  2. for (int i = 0; i < n; i++)
  3.   B ← critical section
  4. C
- Code execution
  - A ⇒ once
  - B ⇒ n times
  - C ⇒ once
- Time ⇒ 1 + n + 1 = O(n)

Critical Section Example 2

- Code (for input size n)
  1. A
  2. for (int i = 0; i < n; i++)
  3.   B
  4.   for (int j = 0; j < n; j++)
  5.   C
  6. D
- Code execution
  - A ⇒
  - B ⇒
  - C ⇒
  - D ⇒
- Time ⇒

Critical Section Example 2

- Code (for input size n)

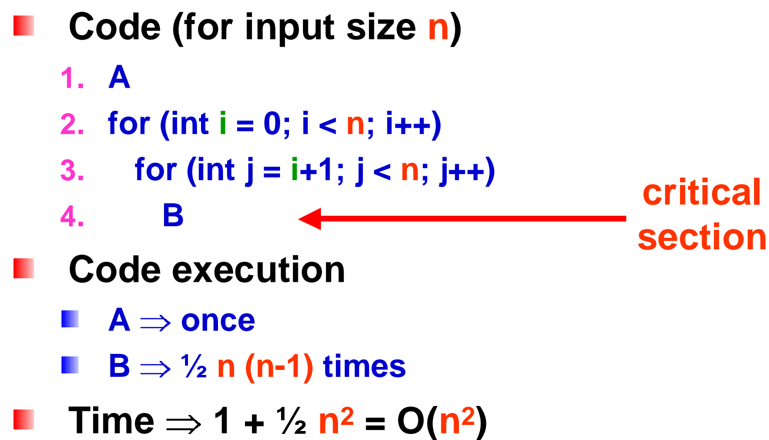


  1. A
  2. for (int i = 0; i < n; i++)
  3.   B
  4.   for (int j = 0; j < n; j++)
  5.   C
  6. D
- Code execution
  - A ⇒ once
  - B ⇒ n times
  - C ⇒ n^2 times
  - D ⇒ once
- Time ⇒ $1 + n + n^2 + 1 = O(n^2)$

Critical Section Example 3

- Code (for input size n)
  1. A
  2. for (int i = 0; i < n; i++)
  3. for (int j = i+1; j < n; j++)
  4. B
- Code execution
  - A ⇒
  - B ⇒
- Time ⇒

Critical Section Example 3



Critical Section Example 4

- Code (for input size n)

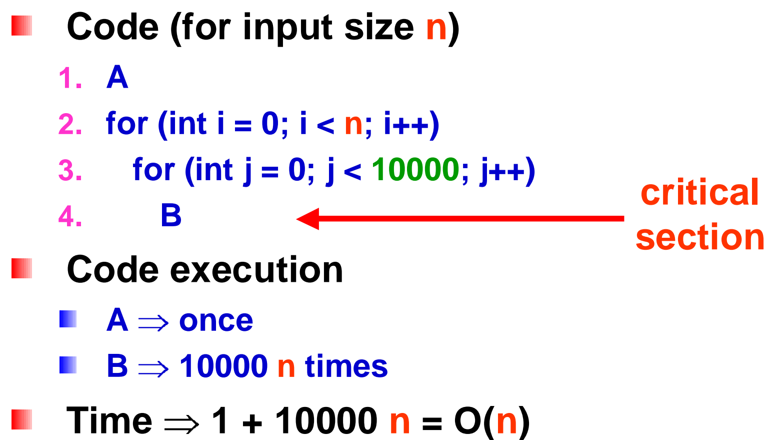
```
1. A
2. for (int i = 0; i < n; i++)
3.   for (int j = 0; j < 10000; j++)
4.     B
```

- Code execution
  - A ⇒
  - B ⇒
- Time ⇒

Critical Section Example 4

- Code (for input size n)

```
1. A
2. for (int i = 0; i < n; i++)
3.   for (int j = 0; j < 10000; j++)
4.     B
```



- Code execution
  - A ⇒ once
  - B ⇒ 10000 n times
- Time ⇒ 1 + 10000 n = O(n)

Critical Section Example 5

- Code (for input size n)

```
1. for (int i = 0; i < n; i++)
2.    for (int j = 0; j < n; j++)
3.       A
4. for (int i = 0; i < n; i++)
5.    for (int j = 0; j < n; j++)
6.       B
```

- Code execution
  - A ⇒
  - B ⇒
- Time ⇒

Critical Section Example 5

- Code (for input size n)

```
1. for (int i = 0; i < n; i++)
2.    for (int j = 0; j < n; j++)
3.       A
4. for (int i = 0; i < n; i++)
5.    for (int j = 0; j < n; j++)
6.       B
```


critical sections

- Code execution
  - A ⇒ n^2 times
  - B ⇒ n^2 times
- Time ⇒ $n^2 + n^2 = O(n^2)$

Critical Section Example 6

- Code (for input size n)
  1. i = 1
  2. while (i < n)
  3.   A
  4.   i = 2 × i
  5. B
- Code execution
  - A ⇒
  - B ⇒
- Time ⇒

Critical Section Example 6

- Code (for input size n)
  1. i = 1
  2. while (i < n)
  3.   A
  4.   i = 2 × i
  5. B

critical section

- Code execution
  - A ⇒ log(n) times
  - B ⇒ 1 times
- Time ⇒ log(n) + 1 = O(log(n))

Critical Section Example 7

- Code (for input size n)

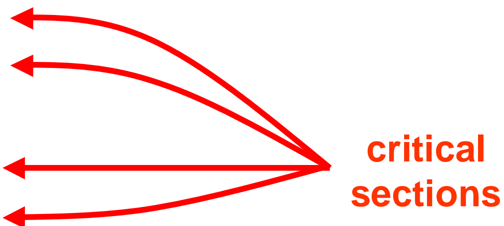
```
1. DoWork (int n)
2.   if (n == 1)
3.     A
4.   else
5.     DoWork(n/2)
6.     DoWork(n/2)
```

- Code execution
  - A $\Rightarrow$
  - DoWork(n/2) $\Rightarrow$
- Time(1) $\Rightarrow$ Time(n) =

Critical Section Example 7

- Code (for input size n)

```
1. DoWork (int n)
2.   if (n == 1)
3.     A
4.   else
5.     DoWork(n/2)
6.     DoWork(n/2)
```



- Code execution
  - A $\Rightarrow$ 1 times
  - DoWork(n/2) $\Rightarrow$ 2 times
- Time(1) $\Rightarrow$ 1   Time(n) = 2 $\times$ Time(n/2) + 1