TREES

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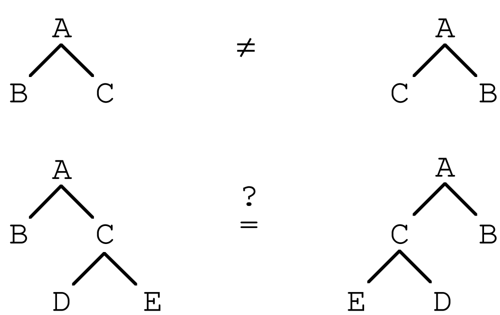
TREE DEFINITION

- TREE ≡ a branching structure between nodes
- A finite set T of one or more nodes such that:
  1. one element of the set is distinguished, ROOT(T)
  2. the remaining nodes of T are partitioned into $m \geq 0$ disjoint sets $T_1, T_2, \ldots T_m$ and each of these sets is in turn a tree.
     - trees $T_1, T_2, \ldots T_m$ are the *subtrees* of the root
- Recursive definition – easy to prove theorems about properties of trees.

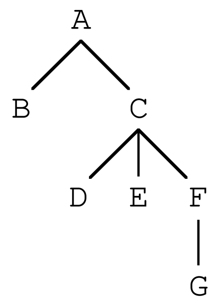
  Ex: prove true for 1 node
  assume true for n nodes
  prove true for n+1 nodes
- ORDERED TREE ≡ if the relative order of the subtrees $T_1, T_2, \ldots T_m$ is important
- ORIENTED TREE ≡ order is not important



ordered

- Computer representation ⇒ ordered!

TERMINOLOGY



- Counterintuitive!

- DEGREE ≡ number of subtrees of a node
- Terminal node ≡ *leaf* ≡ degree 0
- BRANCH NODE ≡ non-terminal node

- Root is the *father* of the roots of its subtrees
- Roots of subtrees of a node are *brothers*
- Roots of subtrees of a node are *sons* of the node
- The root of the tree has no father!
- A is an *ancestor* of C, E, G, …
- G is a *descendant* of A

```
level(X) ≡ if father(X)=Ω then 0
           else 1+level(father(X));

Ex: level(G) = 1+level(F)
                 1+level(C)
                   1+level(A)
                     0
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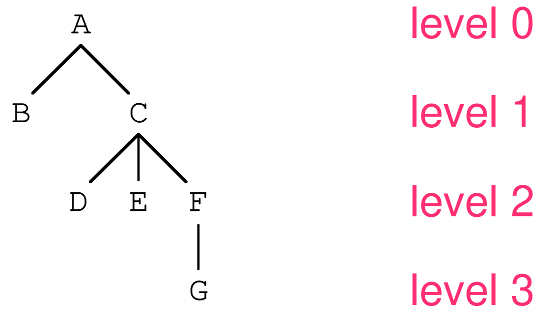

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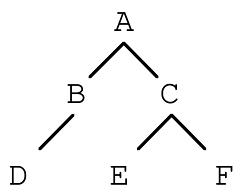
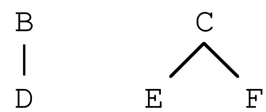
FORESTS AND BINARY TREES

- FOREST ≡ a set (usually ordered) of 0 or more disjoint trees,
  or equivalently:
  the nodes of a tree excluding the root



has the forest



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- Is a binary tree a special case of a tree?

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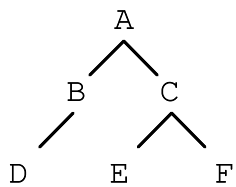

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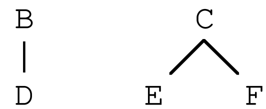

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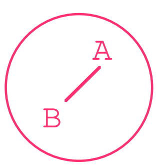
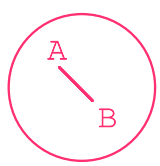
NO! An entirely different concept



1 and



2 are different binary trees

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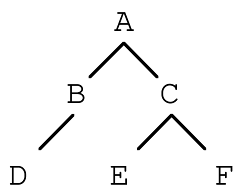

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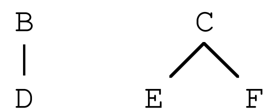

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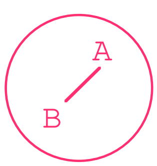
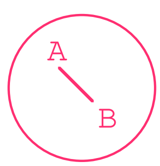


1

and



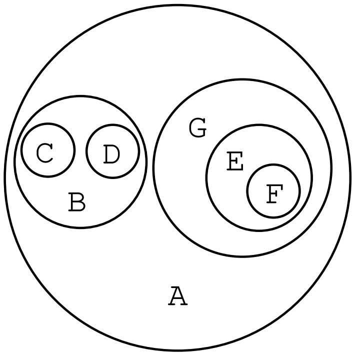
2

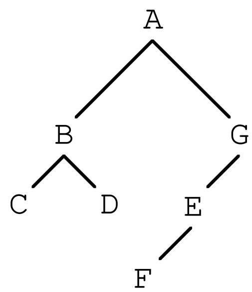
are different binary trees

1 has an empty right subtree
2 has an empty left subtree
But as 'trees' 1 and 2 are identical!

OTHER REPRESENTATIONS OF TREES

- Nested sets (also known as ‘bubble diagrams’)





- Nested parentheses

Tree (root subtree_1 subtree_2 ... subtree_n)

```
(A (B (C) (D)) (G (E (F))))
```

Binary tree (root left right)

```
(A (B (C () ()) (D () ()))
   (G (E (F () ()) ()) ()))
```

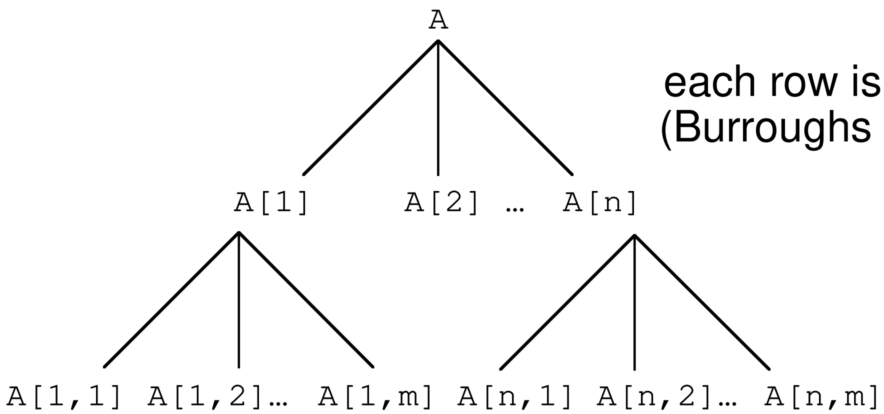
- Indentation

```
A
   B
      C
      D
   G
      E
         F
```

- Dewey decimal notation: `2.1 2.2.2 2.3.4.5`

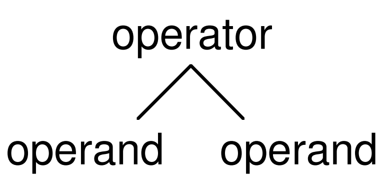
APPLICATIONS

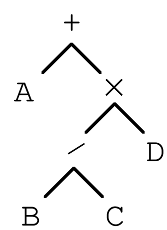
- Segmentation of large rectangular arrays – `A[n,m]`



each row is a segment (Burroughs computers)

- Algebraic formulas





`A+((B⁄C)×D)`

1. no need for parentheses
   - but `A−B+C = (A−B)+C`
     `≠ A−(B+C)`

2. code generation

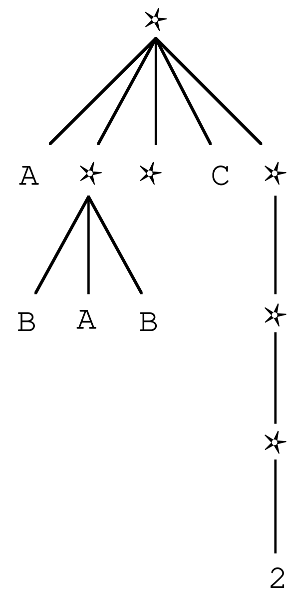
```
LW    1,A
LW    2,B
DW    2,C
MW    2,D
AW    2,1
```

LISTs (with a capital L!)

- LIST ≡ a finite sequence of 0 or more atoms or LISTS

`L=(A,(B,A,B),(),C,(((2))))`

`()` ≡ empty list



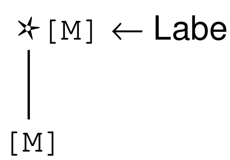
- Index notation:

```
L[2]=(B,A,B)
L[2,1]=B
L[5,2]
L[5,1,1]
```

- Differences between LISTS and trees:

  1. no data appears in the nodes representing LISTS - i.e., ✫

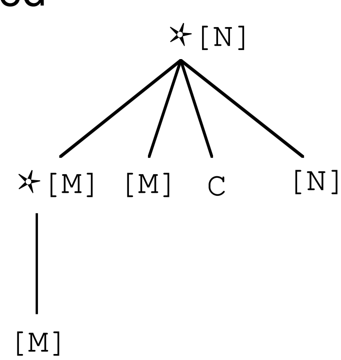
  2. LISTS may be recursive   `M=(M)`



  3. LISTS may overlap (i.e., need not be disjoint)

     - equivalently, subtrees may be shared

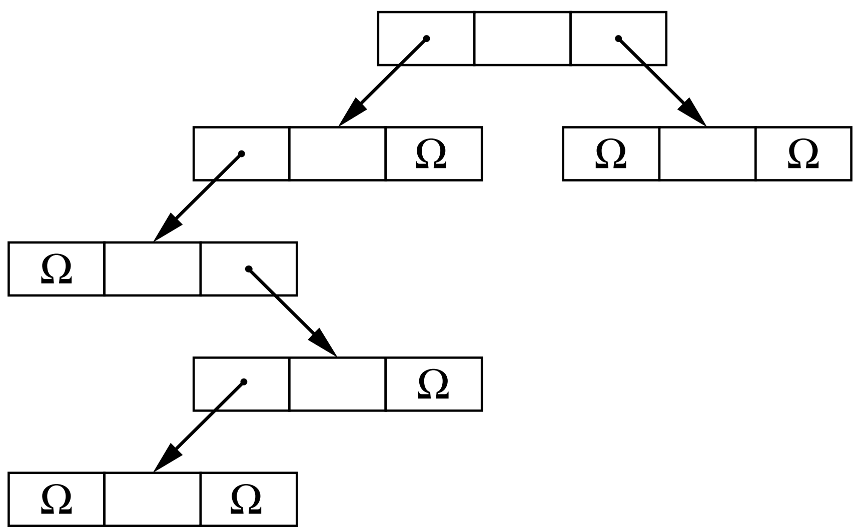
`N=(M,M,C,N)`



TRAVERSING BINARY TREES

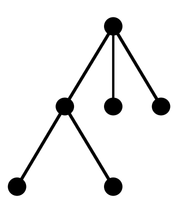
• Representation

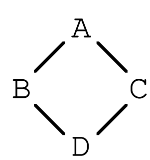
LLINK	INFO	RLINK



• Applications:

1. code generation in compilers
2. game trees in artificial intelligence
3. detect if a structure is really a tree
   - TREE ≡ one path from each node to another node (unlike graph)
   - no cycles



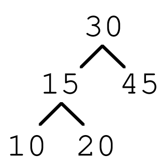


ABD and ACD

TRAVERSAL ORDERS

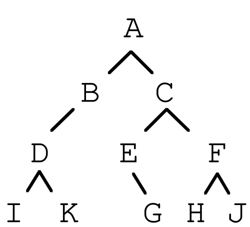
1. Preorder ≡ root, left subtree, right subtree
   - depth-first search
2. Inorder ≡ left subtree, root, right subtree
   - binary search tree
3. Postorder ≡ left subtree, right subtree, root
   - code generation

- Binary search tree: left < root < right



inorder yields `10 15 20 30 45`

- Ex:



preorder =

inorder =

postorder =

- Inorder traversal requires a stack to go back up the tree:

```
D
B
A
```

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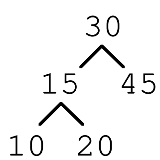

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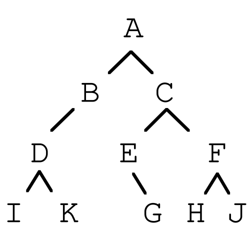

preorder = A B D I K C E G F H J

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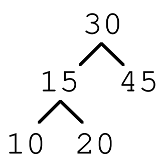

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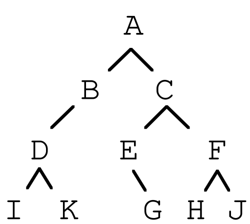

preorder = A B D I K C E G F H J

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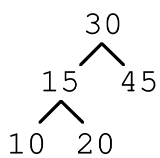

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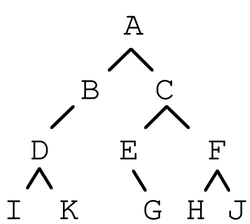

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inorder = I D K B A E G C H F J

postorder = I K D B G E H J F C A

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D

B

A

INORDER TRAVERSAL ALGORITHM

```
procedure inorder(tree pointer T);
begin
  stack A;
  tree pointer P;
  A←Ω;
  P←T;
  while not(P=Ω and A=Ω) do
    begin
      if P=Ω then
        begin
          P⇐A;            /* Pop the stack */
          visit(ROOT(P));
          P←RLINK(P);
        end
      else
        begin
          A⇐P;            /* Push on the stack */
          P←LLINK(P);
        end;
    end;
end;
```

Using recursion:

```
procedure inorder(tree pointer T);
begin
  if T=Ω then return
    else
      begin
        inorder(LLINK(T));
        visit(ROOT(T));
        inorder(RLINK(T));
      end;
end;
```

THREADED BINARY TREES

- Binary tree representation has too many Ω links
- Use 1-bit tag fields to indicate presence of a link
- If Ω link, then use field to store links to other parts of the structure to aid the traversal of the tree

Unthreaded:	Threaded:
LLINK(p) = Ω	LTAG(p) = 0, LLINK(p) = \$p = inorder predecessor of p
LLINK(p) = q $\neq \Omega$	LTAG(p) = 1, LLINK(p) = q
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LLINK	LTAG	INFO	RTAG	RLINK

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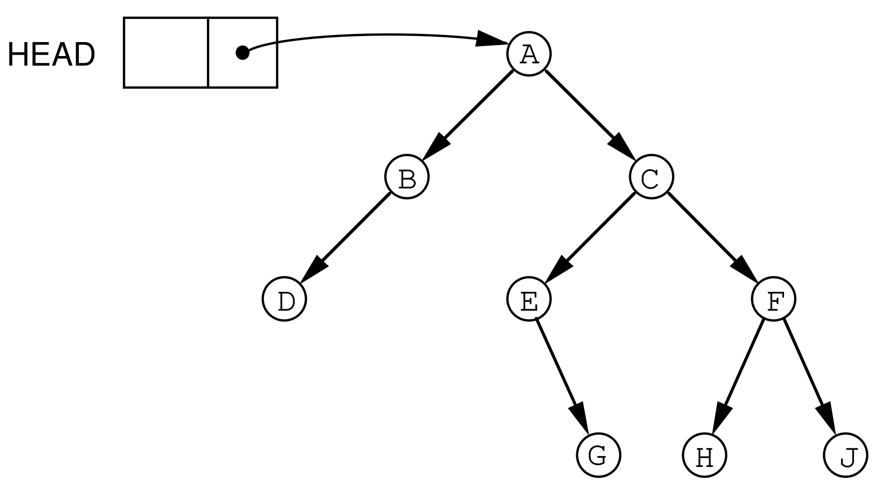

- If address of ROOT(T) < address of left and right sons, then don't need the TAG fields
- Threads will point to lower addresses!

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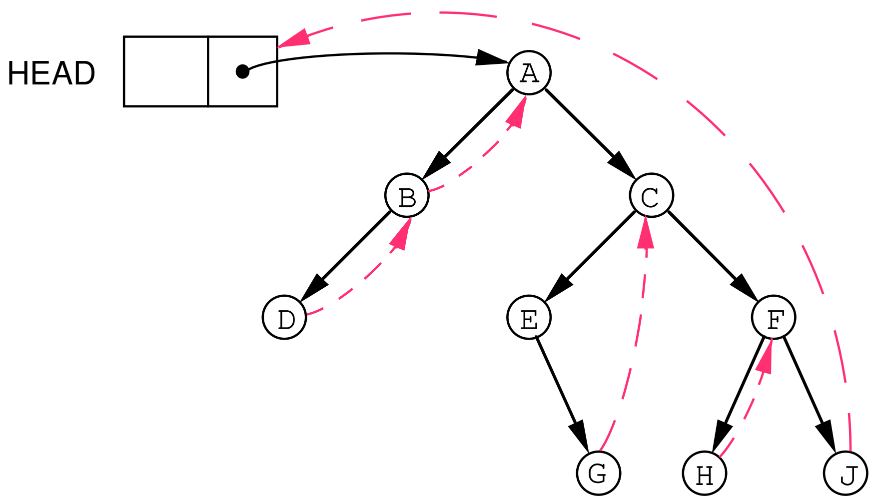

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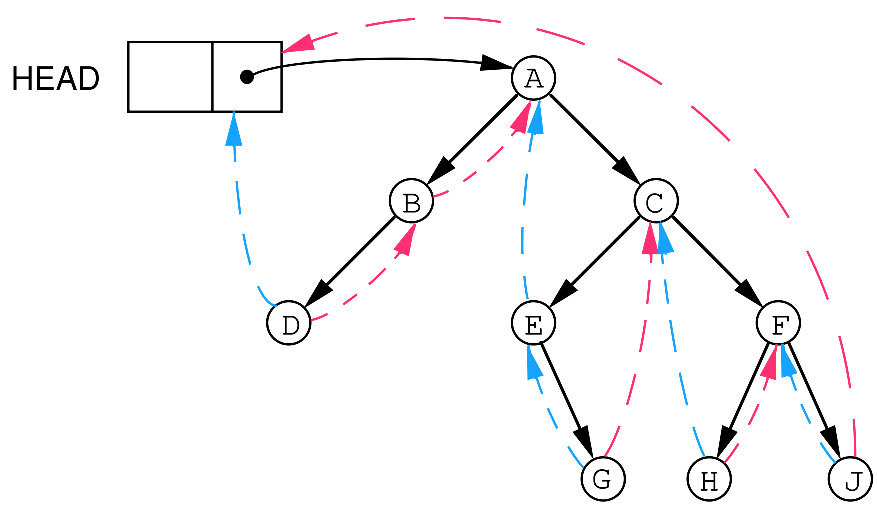

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OPERATIONS ON THREADED BINARY TREES

• Find the inorder successor of node P (P$)

```
1. Q←RLINK(P);      /* right thread points to P$ */
2. if RTAG(P)=1 then
     begin          /* not a thread */
       while LTAG(Q)=1 do Q←LLINK(Q);
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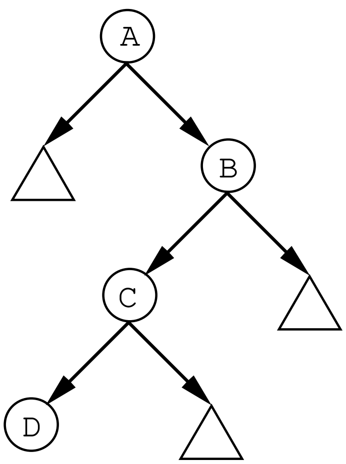
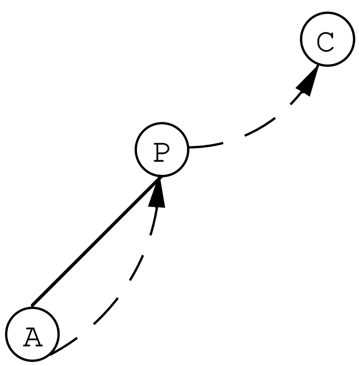
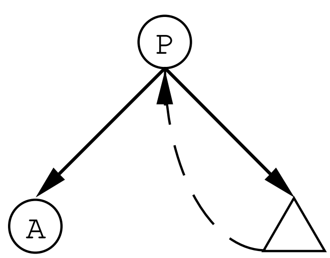


• Insert node Q as the right subtree of node P






```
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     RLINK(P)←Q;             RTAG(P)←1;

     LLINK(Q)←P;             LTAG(Q)←0;

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```

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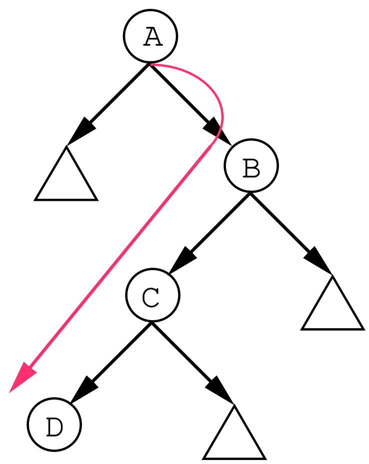

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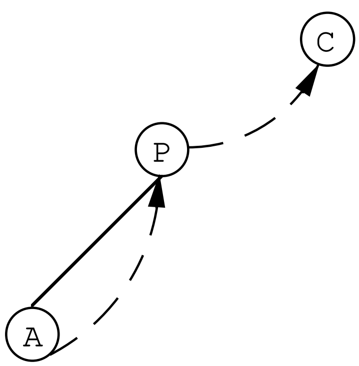

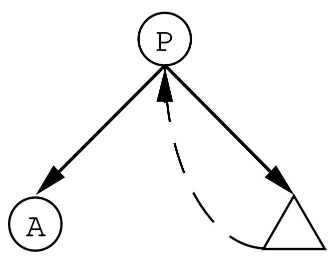


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OPERATIONS ON THREADED BINARY TREES

- Find the inorder successor of node P (P$)

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1. Q←RLINK(P);      /* right thread points to P$ */
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     begin          /* not a thread */
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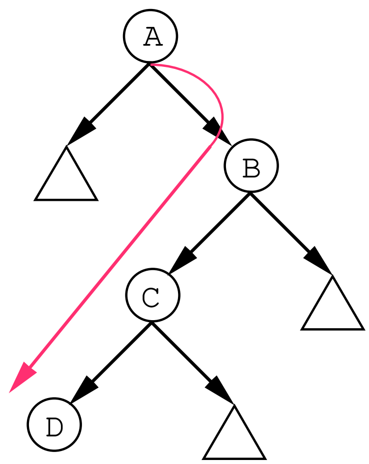
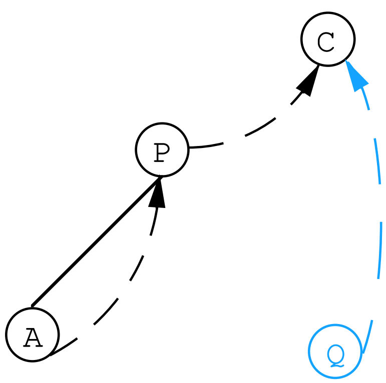
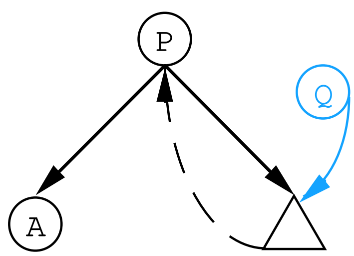


- Insert node Q as the right subtree of node P






```
1.   RLINK(Q)←RLINK(P);      RTAG(Q)←RTAG(P);

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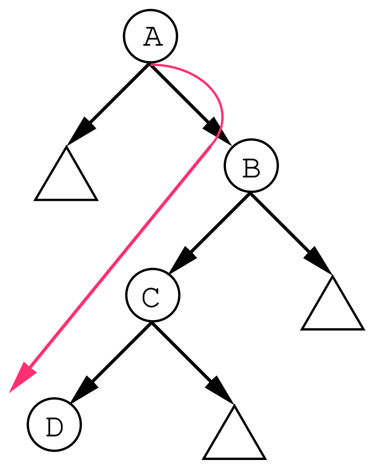

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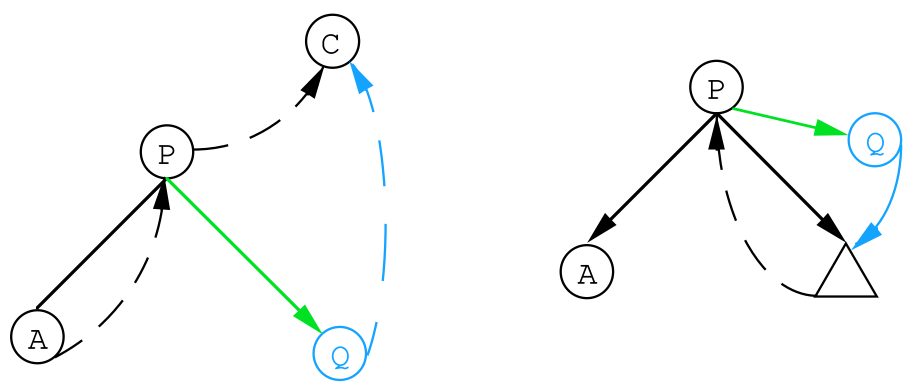


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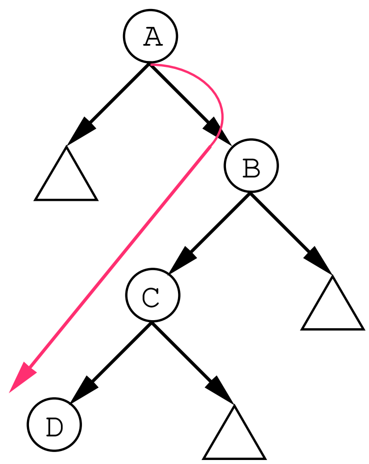

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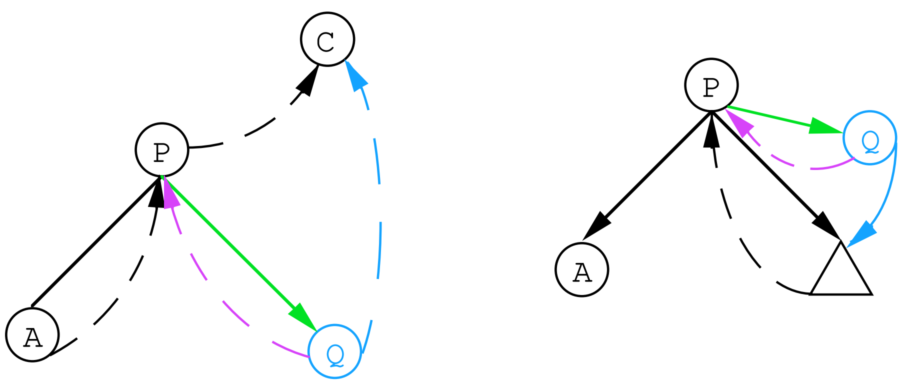


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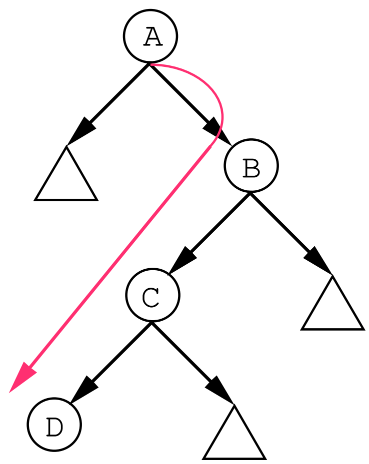

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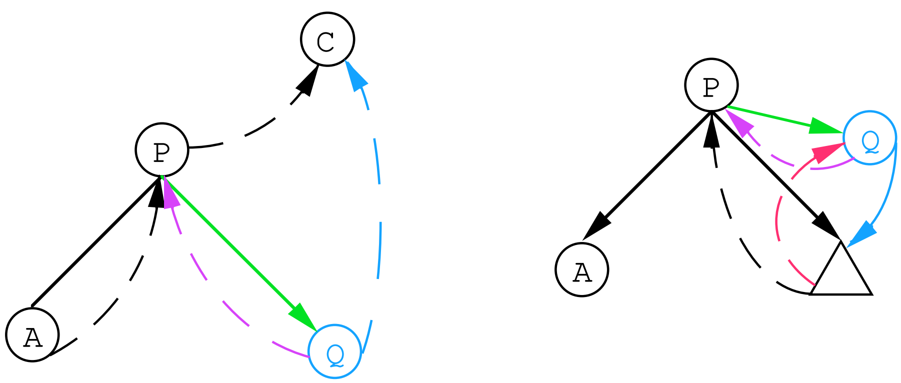


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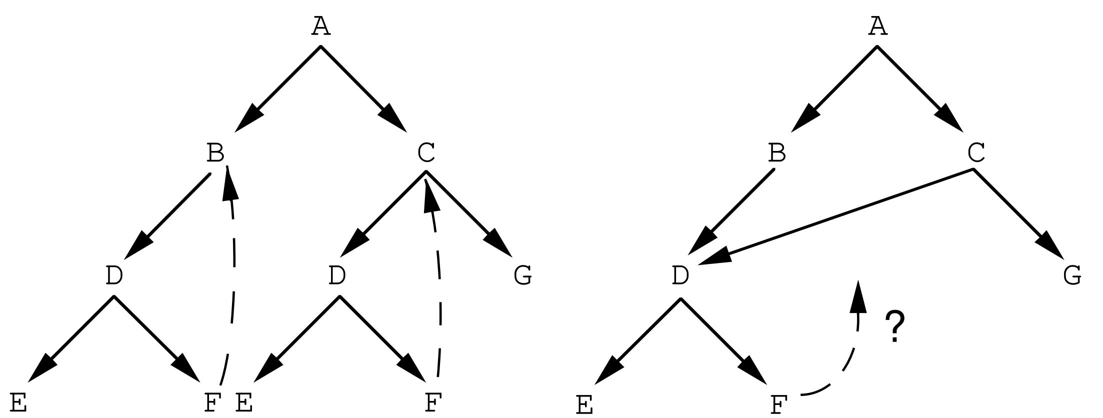
SUMMARY OF THREADING

1. Advantages
   - no need for a stack for traversal
   - will not run out of memory during inorder traversal
   - can find inorder successor of any node without having to traverse the entire tree

2. Disadvantages
   - insertion and deletion of nodes is slower
   - can't share common subtrees in the threaded representation

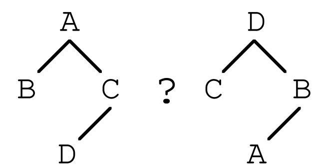
   Ex: two choices for the inorder successor of F



3. Right-threaded trees
   - inorder algorithms make little use of left threads
   - 'LTAG(P)=1' test can be replaced by 'LLINK(P)=Ω' test

PRINCIPLES OF RECURSION

- Two binary trees T1 and T2 are said to be *similar* if they have the same shape or structure
- Formally:
  1. they are both empty *or*
  2. they are both non-empty and their left and right subtrees respectively are similar

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- Will similar work?

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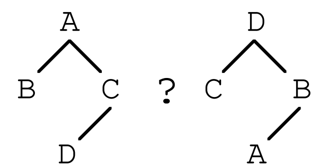


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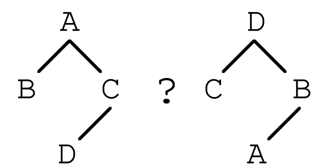


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- Simplifying:

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A and B =  if A then B          A or B =
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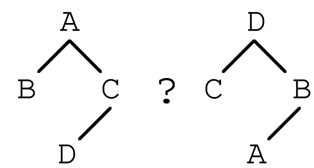


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```
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    if empty(T1) then
      if empty(T2) then T
      else F
    else if empty(T2) then F
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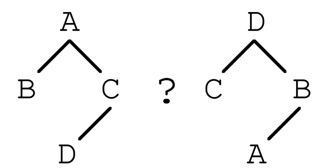


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A and B =  if A then B          A or B =  if A then T
           else F                         else B
```

```
  similar(T1,T2) =
    if empty(T1) then empty(T2)
     [if empty(T2) then T
     else F]
    else if empty(T2) then F                     and
    else [if] similar(left(T1),left(T2)) [then]
              similar(right(T1),right(T2))
   [else F ;]
```

EQUIVALENCE OF BINARY TREES

- Two binary trees T1 and T2 are said to be *equivalent* if they are similar *and* corresponding nodes contain the same information


```
equivalent(T1,T2) =
  if empty(T1) and empty(T2) then T
  else if empty(T1) or empty(T2) then F
  else   root(T1)=root(T2) and
         equivalent(left(T1),left(T2)) and
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EQUIVALENCE OF BINARY TREES

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NO! we are dealing with binary trees and the left subtree of `C` is not the same in the two cases

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```

RECURSION SUMMARY

- Avoids having to use an explicit stack in the algorithm
- Problem formulation is analogous to induction
- Base case, inductive case

- Ex: Factorial
  $n! = n \cdot (n-1)\,!$

```
fact(n) = if n=0 then 1
          else n*fact(n-1);
```

The result is obtained by peeling one's way back along the stack

```
fact(3) = 3*fact(2)
           2*fact(1)
             1*fact(0)
               1
        = 6
```

Using an accumulator variable and a call `fact2(n,1)`:

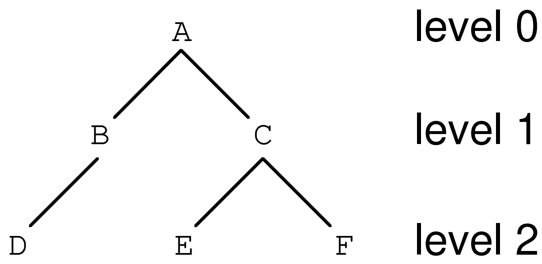
```
fact2(n,total) = if n=0 then total
                 else fact2(n-1,n*total);
```

Solution is iterative

- Recursion implemented on computer using stack instructions.
- Dec-system 10: `PUSH, POP, PUSHJ, POPJ`
- Stack pointer format: (count, address)
- Can simulate stack if no stack instructions

COMPLETE BINARY TREES

When a binary tree is reasonably *complete* (most Ω links are at the highest level), use a sequential storage allocation scheme so that links become unnecessary



1	A
2	B
3	C
4	D
5	Ω
6	E
7	F

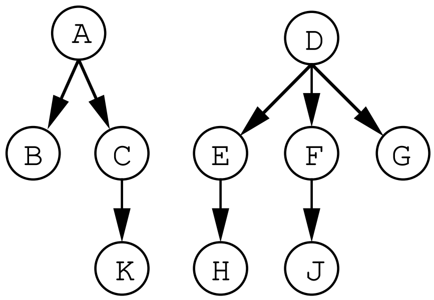
- If n is the highest level at which a node is found, then at most $2^{n+1} - 1$ words are needed

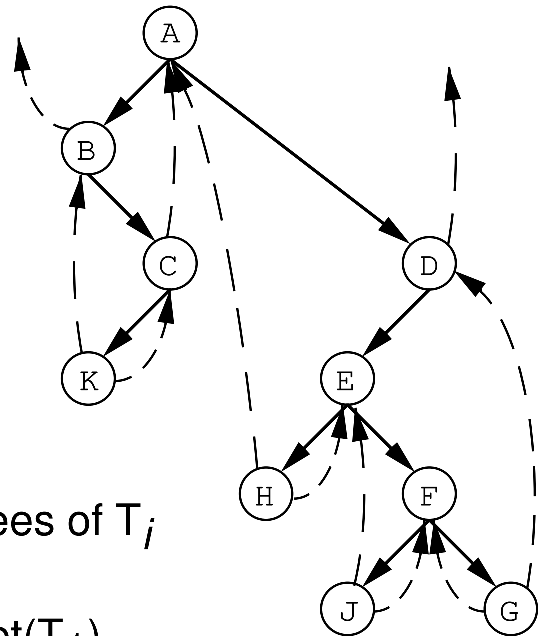
- Storage allocation method:
  1. root has address 1
  2. left son of x has address $2 * \text{address}(x)$
  3. right son of x has address $2 * \text{address}(x) + 1$

- When should a complete binary tree be used?
  n = highest level of the tree at which a node is found
  x = # of nodes in tree
  3 words per node (left link, right link, info)
  use a complete binary tree when $x > (2^{n+1} - 1) / 3$

FORESTS

- A *forest* is an ordered set of 0 or more trees
- There exists a *natural correspondence* between forests and binary trees





- Rigorous definition of B(F)
  $F = (T_1, T_2, \ldots, T_n)$
  $T_{i,1}, T_{i,2}, \ldots, T_{i,m}$ are subtrees of T_i
  1. If $n = 0$, B(F) is empty
  2. If $n > 0$, root of B(F) is root(T_1)
     left subtree of B(F) is $B(T_{1,1}, T_{1,2}, \ldots, T_{1,m})$
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- Traversal of forests

preorder:
1. visit root of first tree
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3. traverse remaining subtrees in preorder

postorder:
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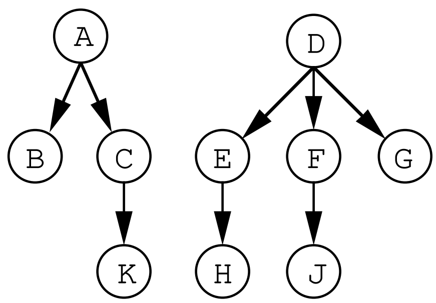

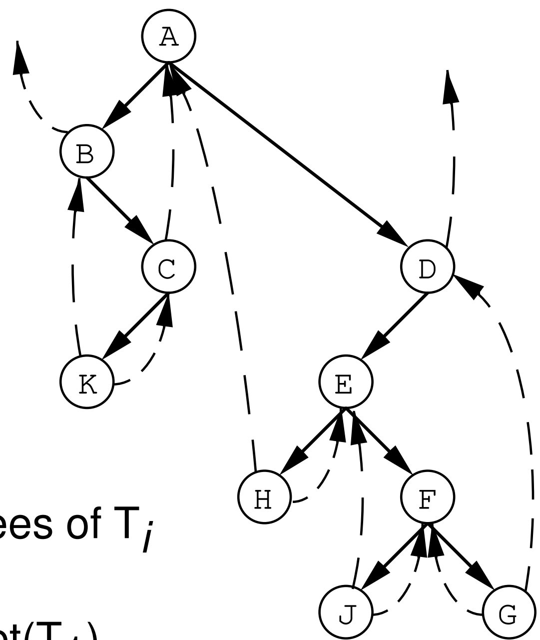

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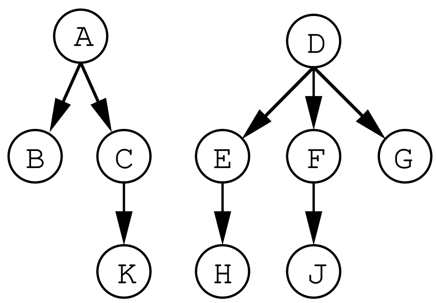

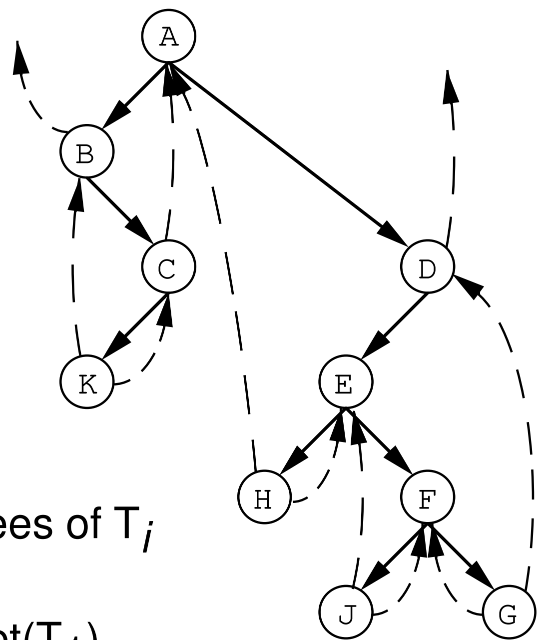

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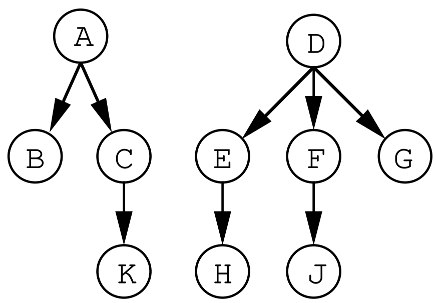

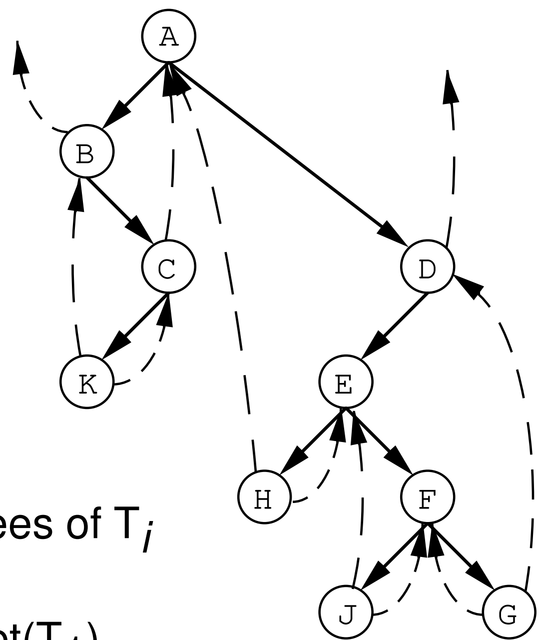

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≡ inorder of binary tree

EQUIVALENCE RELATION

- Given: relations as to what is equivalent to what ($a \equiv b$)
- Goal: is $x \equiv y$?

- Formal definition of an *equivalence relation*
  1. if $x \equiv y$ and $y \equiv z$ then $x \equiv z$ (transitivity)
  2. if $x \equiv y$ then $y \equiv x$ (symmetry)
  3. $x \equiv x$ (reflexivity)

- Ex: $S = \{1 .. 9\}$

  $1 \equiv 5$ $6 \equiv 8$ $7 \equiv 2$ $9 \equiv 8$ $3 \equiv 7$ $4 \equiv 2$ $9 \equiv 3$

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- Partitions S into disjoint subsets or *equivalence classes*
- Two elements equivalent iff they belong to same class
- What are the equivalence classes in this example?

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  $\{1,5\}$ and $\{2,3,4,6,7,8,9\}$

ALGORITHM

- Represent each element as a node in forest of trees
- Trees consist only of father links (nil at roots)
- Each (nonredundant) relation merges two trees into one
- Basic strategy:

```
for each relation a≡b do
  begin
    find root node r of tree containing a; /* Find step */
    find root node s of tree containing b;
    if they differ, merge the two trees; /* Union step */
  end;
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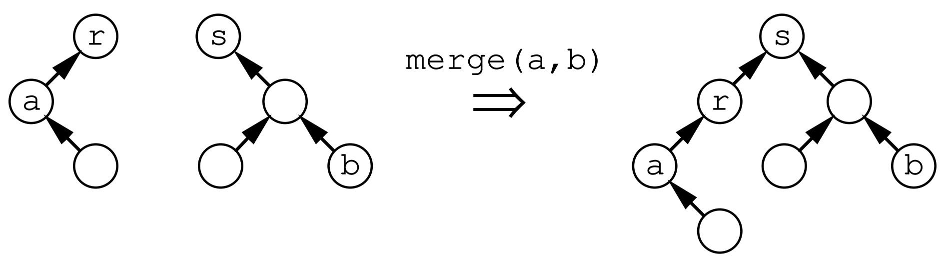
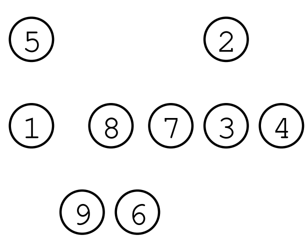


- Algorithm (also known as *union-find*):

```
for every element i do father(i)←Ω
while input_not_exhausted do
  begin
    get_pair(a,b);
    while father(a)≠Ω do a←father(a);
    while father(b)≠Ω do b←father(b);
    if (a≠b) then father(a)←b;
  end;
```




```
father(k):
       k: 1 2 3 4 5 6 7 8 9
```

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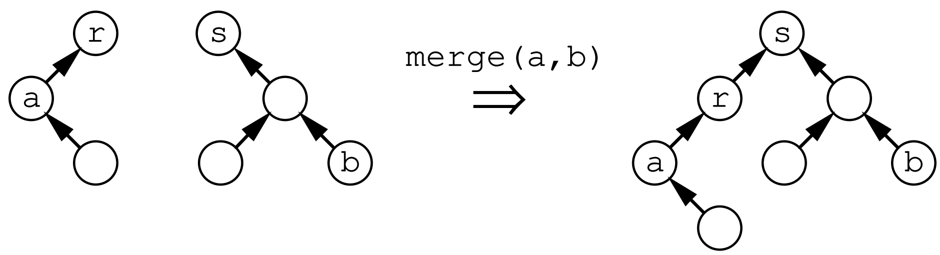

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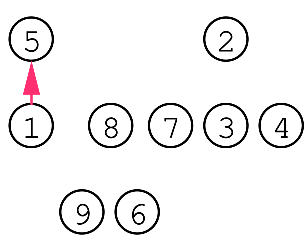
1≡5




```
father(k): 5
        k: 1 2 3 4 5 6 7 8 9
```

ALGORITHM

- Represent each element as a node in forest of trees
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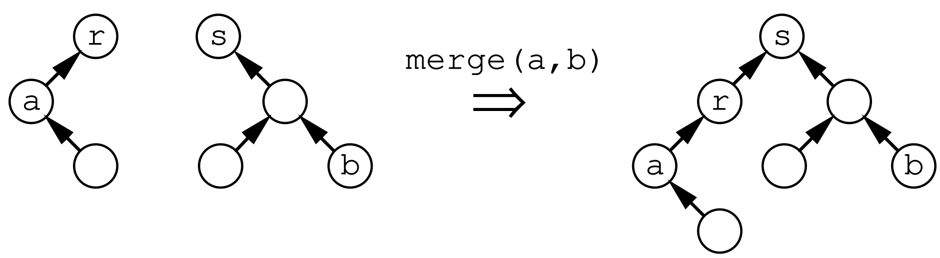

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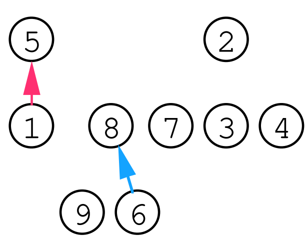
1≡5
6≡8



father(k):	5					8			
k:	1	2	3	4	5	6	7	8	9

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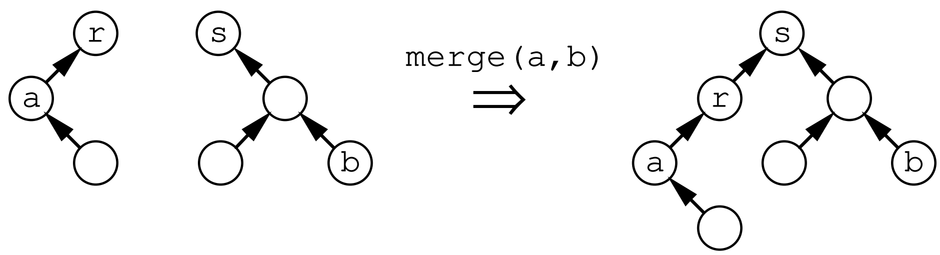

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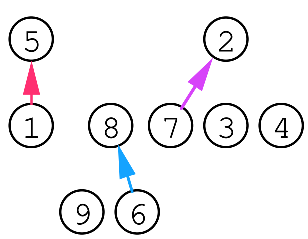
1≡5
6≡8
7≡2



father(k):	5					8	2		
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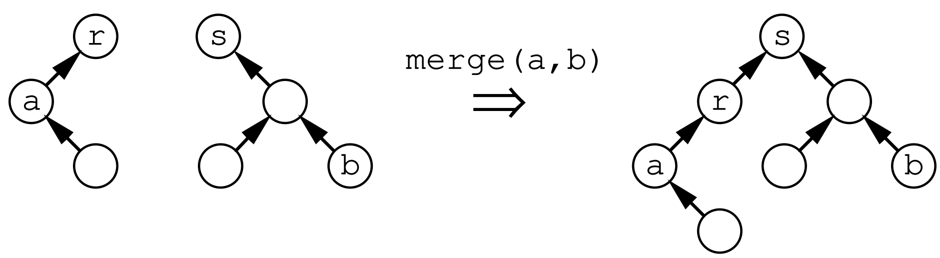

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6≡8
7≡2
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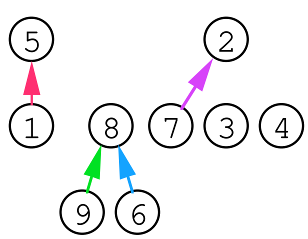

father(k):	5					8	2		8
k:	1	2	3	4	5	6	7	8	9

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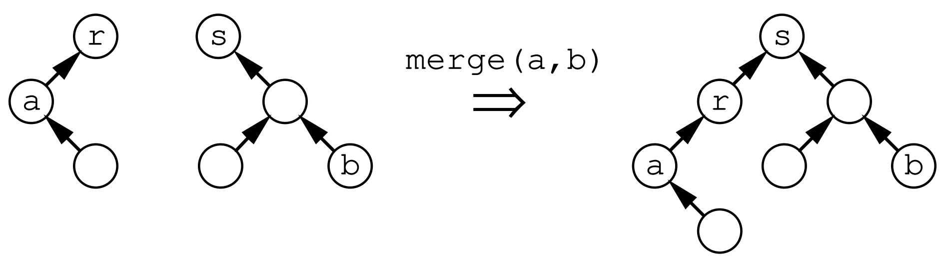

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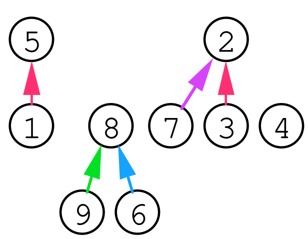
9≡8
3≡7



father(k):	5		2			8	2		8
k:	1	2	3	4	5	6	7	8	9

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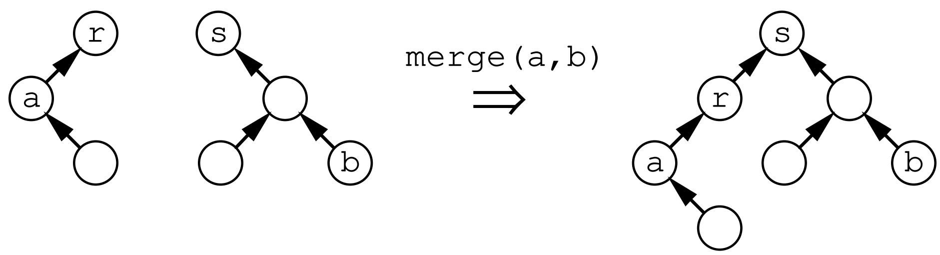

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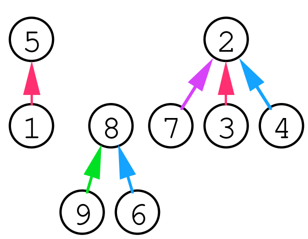

father(k):	5		2	2		8	2		8
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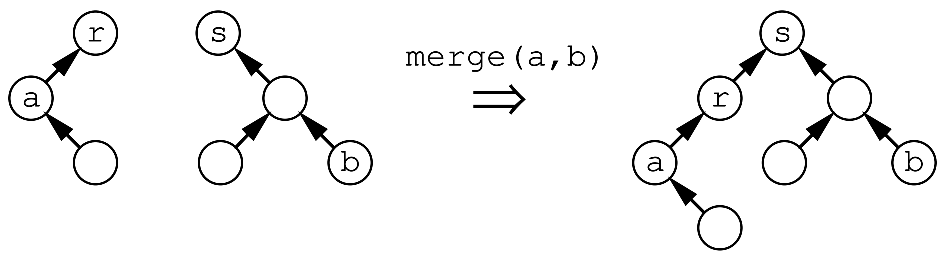

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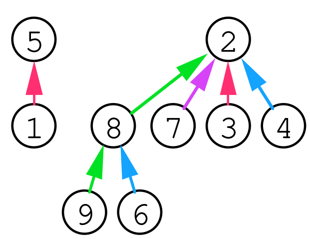
7≡2
9≡8
3≡7
4≡2
9≡3



father(k):	5		2	2		8	2	2	8
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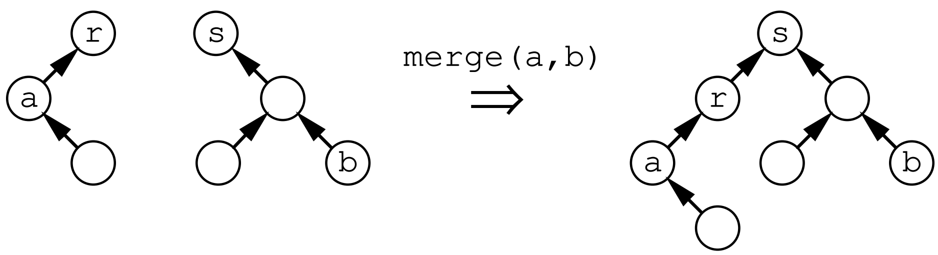

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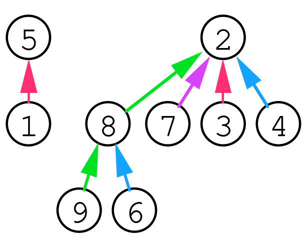

father(k):	5		2	2		8	2	2	8
k:	1	2	3	4	5	6	7	8	9

- More efficient with *path compression* and *weight balancing*
- Execution time “almost linear” (inverse of Ackermann function)