CMSC 858S, Spring 2007: HW 5, due on April 11th (Wed.) at the start of class

Notes: Please work on this with your group-mate(s); just submit *one* writeup per group. Consulting other sources (including the Web) is not allowed. Write your solutions neatly; if you are able to make partial progress by making some additional assumptions, then state these assumptions clearly and submit your partial solution. The problem marked (*) may be more difficult than the others.

1. Suppose you are given an undirected graph $G = (V, E)$ where the maximum degree of any vertex is Δ. (Recall that the degree of a vertex is the number of edges incident upon it.) Let $d(u)$ be the degree of vertex u; given a partition of V into two sets A and B, let $d_A(u)$ and $d_B(u)$ denote the number of neighbors of u in A and the number of neighbors of u in B, respectively. Show that there is a constant $C > 0$ and a partition of V into two sets A and B such that for all u,

$$d_A(u) \leq d(u)/2 + C\sqrt{\Delta \log \Delta} \text{ and } d_B(u) \leq d(u)/2 + C\sqrt{\Delta \log \Delta}.$$

(10 points)

2. Show that there is a constant $a > 0$ such that the following holds. We have an arbitrary graph $G = (V, E)$ with maximum degree Δ. Each vertex v also has a list of colors L_v; we want to color each vertex v with some color from L_v, so that we get a proper coloring (i.e., no two adjacent vertices get the same color). Prove that this is possible if the following holds: there is a non-negative value $b_{v,c}$ for all vertices v and all colors $c \in L_v$, such that:

- $\forall v, \ \sum_{c \in L_v} b_{v,c} = 1$; and
- $\forall (u, v) \in E, \ \sum_{c \in L_u \cap L_v} b_{u,c} \cdot b_{v,c} \leq a/\Delta$.

Try to get as large a value for the constant a as you can; however, your score for this problem will not depend on what constant a you get. **(5 points)**

3(*). Show that there is an integer constant $a > 0$ such that the following holds. We have an arbitrary graph $G = (V, E)$ with maximum degree Δ. Show that we can give a color from $\{1, 2, \ldots, a\Delta\}$ to each *edge* of G, so that the following hold:

- no two edges that share an end-point get the same color;
- no *even-length* cycle has only two colors given to its set of edges.

(**Hint:** Do an appropriate random construction. In addition to certain other bad events, associate one bad event $A_{e,f}$ with every pair of edges e and f that share an end-point. Use the *asymmetric* version of the Local Lemma; thus, in particular, you need to come up with suitable values $x_{e,f}$. Take $x_{e,f}$ a constant times $\Pr[A_{e,f}]$, and come up with suitable choices for the other parameters you need to define for the LLL.) **(15 points)**