Notes: Please work on this with your group-mate(s); just submit one writeup per group. Consulting other sources (including the Web) is not allowed. Write your solutions neatly; if you are able to make partial progress by making some additional assumptions, then state these assumptions clearly and submit your partial solution. The problem marked (*) may be more difficult than the others.

1. Suppose you are given an undirected graph $G = (V, E)$ where the maximum degree of any vertex is $\Delta$. (Recall that the degree of a vertex is the number of edges incident upon it.) Let $d(u)$ be the degree of vertex $u$; given a partition of $V$ into two sets $A$ and $B$, let $d_A(u)$ and $d_B(u)$ denote the number of neighbors of $u$ in $A$ and the number of neighbors of $u$ in $B$, respectively. 

Show that there is a constant $C > 0$ and a partition of $V$ into two sets $A$ and $B$ such that for all $u$,

$$d_A(u) \leq d(u)/2 + C\sqrt{\Delta \log \Delta} \quad \text{and} \quad d_B(u) \leq d(u)/2 + C\sqrt{\Delta \log \Delta}.$$ 

(10 points)

2. Show that there is a constant $a > 0$ such that the following holds. We have an arbitrary graph $G = (V, E)$ with maximum degree $\Delta$. Each vertex $v$ also has a list of colors $L_v$; we want to color each vertex $v$ with some color from $L_v$, so that we get a proper coloring (i.e., no two adjacent vertices get the same color). Prove that this is possible if the following holds: there is a non-negative value $b_{v,c}$ for all vertices $v$ and all colors $c \in L_v$, such that:

- $\forall v, \sum_{c \in L_v} b_{v,c} = 1$; and 
- $\forall (u,v) \in E, \sum_{c \in L_u \cap L_v} b_{u,c} \cdot b_{v,c} \leq a/\Delta$.

Try to get as large a value for the constant $a$ as you can; however, your score for this problem will not depend on what constant $a$ you get. (5 points)

3(*). Show that there is an integer constant $a > 0$ such that the following holds. We have an arbitrary graph $G = (V, E)$ with maximum degree $\Delta$. Show that we can give a color from $\{1, 2, \ldots, a\Delta\}$ to each edge of $G$, so that the following hold:

- no two edges that share an end-point get the same color;
- no even-length cycle has only two colors given to its set of edges.

(Hint: Do an appropriate random construction. In addition to certain other bad events, associate one bad event $A_{e,f}$ with every pair of edges $e$ and $f$ that share an end-point. Use the asymmetric version of the Local Lemma; thus, in particular, you need to come up with suitable values $x_{e,f}$. Take $x_{e,f}$ a constant times $\Pr[A_{e,f}]$, and come up with suitable choices for the other parameters you need to define for the LLL.) (15 points)