CMSC 132: Object-Oriented Programming II

Algorithmic Complexity II

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Overview

- Critical sections
- Comparing complexity
- Types of complexity analysis
Analyzing Algorithms

Goal

- Find asymptotic complexity of algorithm

Approach

- Ignore less frequently executed parts of algorithm
- Find critical section of algorithm
- Determine how many times critical section is executed as function of problem size
Critical Section of Algorithm

Heart of algorithm

Dominates overall execution time

Characteristics
- Operation central to functioning of program
- Contained inside deeply nested loops
- Executed as often as any other part of algorithm

Sources
- Loops
- Recursion
Critical Section Example 1

Code (for input size $n$)

1. A
2. for (int $i = 0; i < n; i++$)
3. B
4. C

critical section

Code execution

- A $\Rightarrow$ once
- B $\Rightarrow$ $n$ times
- C $\Rightarrow$ once

Time $\Rightarrow 1 + n + 1 = O(n)$
Critical Section Example 2

Code (for input size \( n \))

1. A
2. for (int i = 0; i < n; i++)
3. B
4. for (int j = 0; j < n; j++)
5. C
6. D

Code execution

- A \( \Rightarrow \) once
- B \( \Rightarrow \) \( n \) times
- C \( \Rightarrow \) \( n^2 \) times
- D \( \Rightarrow \) once

Time \( \Rightarrow 1 + n + n^2 + 1 = O(n^2) \)
Critical Section Example 3

Code (for input size $n$)

1. A
2. for (int $i = 0; i < n; i++$)
3. for (int $j = i+1; j < n; j++$)
4. B

Code execution

- A $\Rightarrow$ once
- B $\Rightarrow \frac{1}{2} n (n-1)$ times

Time $\Rightarrow 1 + \frac{1}{2} n^2 = O(n^2)$
Critical Section Example 4

Code (for input size \( n \))

1. A
2. for (int i = 0; i < n; i++)
3. for (int j = 0; j < 10000; j++)
4. B

Code execution

- A \( \Rightarrow \) once
- B \( \Rightarrow 10000 \) \( n \) times

Time \( \Rightarrow 1 + 10000 \) \( n \) = \( O(n) \)
Critical Section Example 5

Code (for input size $n$)

1. for (int i = 0; i < $n$; i++)
2. for (int j = 0; j < $n$; j++)
3. A
4. for (int i = 0; i < $n$; i++)
5. for (int j = 0; j < $n$; j++)
6. B

Code execution

- A $\Rightarrow n^2$ times
- B $\Rightarrow n^2$ times

Time $\Rightarrow n^2 + n^2 = O(n^2)$
Critical Section Example 6

Code (for input size $n$)

1. $i = 1$
2. while ($i < n$)
3. A
4. $i = 2 \times i$
5. B

Code execution

- A $\Rightarrow \log(n)$ times
- B $\Rightarrow$ 1 times

Time $\Rightarrow \log(n) + 1 = O(\log(n))$
Critical Section Example 7

Code (for input size $n$)

1. DoWork (int $n$)
2. if ($n == 1$)
3. A
4. else
5. DoWork($n/2$)
6. DoWork($n/2$)

Code execution

- A $\Rightarrow 1$ times
- DoWork($n/2$) $\Rightarrow 2$ times

Time(1) $\Rightarrow 1$  
Time($n$) = $2 \times$ Time($n/2$) + 1
Recursive Algorithms

Definition

- An algorithm that calls itself

Components of a recursive algorithm

1. Base cases
   - Computation with no recursion
2. Recursive cases
   - Recursive calls
   - Combining recursive results
Recursive Algorithm Example

Code (for input size n)

1. DoWork (int n)
2. if (n == 1)
3. A
4. else
5. DoWork(n/2)
6. DoWork(n/2)

base case
recursive cases
## Asymptotic Complexity Categories

<table>
<thead>
<tr>
<th>Complexity</th>
<th>Name</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>O(1)</td>
<td>Constant</td>
<td>Array access</td>
</tr>
<tr>
<td>O(log(n))</td>
<td>Logarithmic</td>
<td>Binary search</td>
</tr>
<tr>
<td>O(n)</td>
<td>Linear</td>
<td>Largest element</td>
</tr>
<tr>
<td>O(n log(n))</td>
<td>N log N</td>
<td>Optimal sort</td>
</tr>
<tr>
<td>O(n^2)</td>
<td>Quadratic</td>
<td>2D Matrix addition</td>
</tr>
<tr>
<td>O(n^3)</td>
<td>Cubic</td>
<td>2D Matrix multiply</td>
</tr>
<tr>
<td>O(n^k)</td>
<td>Polynomial</td>
<td>Linear programming</td>
</tr>
<tr>
<td>O(k^n)</td>
<td>Exponential</td>
<td>Integer programming</td>
</tr>
</tbody>
</table>

From smallest to largest

For size \( n \), constant \( k > 1 \)
Comparing Complexity

- Compare two algorithms
  - $f(n)$, $g(n)$
- Determine which increases at faster rate
  - As problem size $n$ increases
- Can compare ratio
  - If $\infty$, $f()$ is larger
  - If 0, $g()$ is larger
  - If constant, then same complexity

$$\lim_{n \to \infty} \frac{f(n)}{g(n)}$$
Complexity Comparison Examples

- \( \log(n) \) vs. \( n^{\frac{1}{2}} \)

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} \quad \rightarrow \quad \lim_{n \to \infty} \frac{\log(n)}{n^{\frac{1}{2}}} \quad \rightarrow \quad 0
\]

- \( 1.001^n \) vs. \( n^{1000} \)

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} \quad \rightarrow \quad \lim_{n \to \infty} \frac{1.001^n}{n^{1000}} \quad \rightarrow \quad ??
\]

Not clear, use L’Hopital’s Rule
Additional Complexity Measures

- **Upper bound**
  - Big-O \( \Rightarrow O(...) \)
  - Represents upper bound on # steps

- **Lower bound**
  - Big-Omega \( \Rightarrow \Omega(...) \)
  - Represents lower bound on # steps

- **Combined bound**
  - Big-Theta \( \Rightarrow \Theta(...) \)
  - Represents combined upper/lower bound on # steps
  - Best possible asymptotic solution
2D Matrix Multiplication Example

Problem

\[ C = A \times B \]

Lower bound

\[ \Omega(n^2) \]

Required to examine 2D matrix

Upper bounds

\[ O(n^3) \]

Basic algorithm

\[ O(n^{2.807}) \]

Strassen’s algorithm (1969)

\[ O(n^{2.376}) \]

Coppersmith & Winograd (1987)

Improvements still possible (open problem)

Since upper & lower bounds do not match
### Additional Complexity Categories

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>Deterministic polynomial time</td>
</tr>
<tr>
<td>NP</td>
<td>Nondeterministic polynomial time</td>
</tr>
<tr>
<td>PSPACE</td>
<td>Polynomial space</td>
</tr>
<tr>
<td>EXPSPACE</td>
<td>Exponential space</td>
</tr>
<tr>
<td>Decidable</td>
<td>Can be solved by finite algorithm</td>
</tr>
<tr>
<td>Undecidable</td>
<td>Not solvable by finite algorithm</td>
</tr>
</tbody>
</table>

If a problem has an algorithm that solves it in time $X$, then the problem is said to be in $X$

- e.g., matrix multiplication is in P
Why do we care?

- If a problem can’t be solved with in P time, then no algorithm can solve *all* cases exactly in a *reasonable* amount of time.
  - But there might be an algorithm that always quickly gives close approximations.
  - Or an algorithm that gives quick exact answers for the cases we care about.

- Issues such as PSPACE vs. EXPSPACE are interesting theoretical issues, and sometime provide practice insight.
NP Time Algorithms

Two ways of thinking about it

First way:
- Given a problem, and a potential answer, can we verify that the answer is correct in polynomial time?

Second way:
- Whenever the algorithm wants, it can clone itself in two
- If either clone finds an answer, the entire system produces an answer
Graph 3-coloring

- Given a graph (vertices and undirected edges)
- Can you find a way to color each vertex either blue, red, or green
- Such that no two vertices connected by an edge have the same color?
Some graphs are hard
3 coloring a graph is in NP

- There exist NP algorithms to 3 color a graph
  - Guess a 3 coloring
  - Verify that the coloring is valid
    - Easy to do in O(n) time

- No one knows if there exists a polynomial time algorithm to find a 3 coloring for an arbitrary graph
  - If you come up with one, even one that runs in time O(n^{100}), you win one million dollars
  - Seriously
Many interesting problems are solvable with an NP algorithm, but not known to be solvable with a P algorithm

- Boolean satisfiability
- Traveling salesman problem (TLP)
- Bin packing

Key to solving many optimization problems

- Most efficient trip routes
- Most efficient schedule for employees
- Most efficient usage of resources
NP Complete problems

Some problems are NP-complete
- They are in NP
- And if a polynomial time algorithm existed for the program
- Then every single last problem in NP could be solved in polynomial time

Almost all problems that are in NP but are not known to be in P are NP-complete
- But not all
P = NP?

Are NP problems solvable in polynomial time?

- Prove \( P = NP \)
  - Show polynomial time solution exists for any NP-complete problem

- Prove \( P \neq NP \)
  - Show no polynomial-time solution possible for some problem in NP
  - The expected answer

The most important open problem in CS

- $1 million prize offered by Clay Math Institute
- Plus front page, NY Times, job offers galore, instant Ph.D. in Computer Science
Algorithmic Complexity Summary

- Asymptotic complexity
  - Fundamental measure of efficiency
  - Independent of implementation & computer platform

- Learned how to
  - Examine program
  - Find critical sections
  - Calculate complexity of algorithm
  - Compare complexity