1. Prove the union bound using Markov’s inequality. \(5\) points

2. Construct a random variable \(X\) with some mean \(\mu\), such that the “first central moment” \(E[|X - \mu|]\) is “small” (say, at most 2) but such that the variance (i.e., second central moment) \(E[(X - \mu)^2]\) can be made arbitrarily large by adjusting some parameter in the definition of \(X\). \(5\) points

3. Recall an issue we discussed in the first class: the assumption that we have a source of unbiased and independent random bits. Here is one simple case of a source \(S\) of somewhat-weak randomness from which we can extract any number of unbiased and independent random bits. Suppose, for some unknown real \(p \in (0, 1)\), that \(S\) outputs bits \(X_1, X_2, X_3, \ldots\) that are independent, but have \(\Pr[X_i = 1] = p\) for all \(i\). Show how to extract unbiased and independent random bits from \(S\). \(10\) points

4. Suppose that in a two-player game, one player chooses some mixed strategy \(p\). Then prove that there exists a pure strategy for the second player that is a best-response to \(p\). \(5\) points

5. You are given a Boolean formula \(\phi\) with \(m\) clauses in conjunctive normal form, with at least 3 literals in each clause. For instance, \(\phi\) could be:

\[
(X_2 \lor \overline{X}_5 \lor \overline{X}_8) \land (X_1 \lor X_5 \lor \overline{X}_6 \lor \overline{X}_8) \land (\overline{X}_2 \lor X_6 \lor \overline{X}_9) \land (X_3 \lor X_4 \lor \overline{X}_7 \lor \overline{X}_{10});
\]

there are \(m = 4\) clauses here, and the underlying Boolean variables are \(X_1, \ldots, X_{10}\).

Given any such \(\phi\), show that there exists an assignment of truth values to the underlying Boolean variables that satisfies at least \(7m/8\) of the given \(m\) clauses. \(5\) points

6. Let \(G = (V, E)\) be a graph with \(n\) vertices and minimum degree \(\delta \geq 2\). Show that there is a partition of \(V\) into two subsets \(A\) and \(B\) such that \(|A| \leq O(n \log(\delta + 1)/\delta)\), and such that each vertex of \(B\) has at least one neighbor in \(A\) and at least one neighbor in \(B\). \(10\) points \(\text{(Hint: Start with a simple random process followed by a carefully-designed alteration.)}\)