CMSC 858C, Spring 2009: Homework 1, due on February 17th (Tuesday) at the start of class.

Notes: Please work on this with your group-mate(s): each group needs to submit only one solution. Consulting other sources (including the Web) is not allowed. Write your solutions **neatly**; if you are able to make partial progress by making some additional assumptions, then state these assumptions clearly and submit your partial solution.

1. Prove the union bound using Markov's inequality. **(5 points)**

2. Construct a random variable X with some mean μ, such that the "first central moment" $\mathbf{E}[|X-\mu|]$ is "small" (say, at most 2) but such that the variance (i.e., second central moment) $\mathbf{E}[(X-\mu)^2]$ can be made arbitrarily large by adjusting some parameter in the definition of X. **(5 points)**

3. Recall an issue we discussed in the first class: the assumption that we have a source of unbiased and independent random bits. Here is one simple case of a source S of somewhat-weak randomness from which we can extract any number of unbiased and independent random bits. Suppose, for some *unknown* real $p \in (0,1)$, that S outputs bits $X_1, X_2, X_3, \ldots$ that are *independent*, but have $\Pr[X_i = 1] = p$ for all i. Show how to extract unbiased and independent random bits from S. (**Hint:** Group the output of S into collections of two bits each.) **(10 points)**

4. Suppose that in a two-player game, one player chooses some mixed strategy p. Then prove that there exists a *pure* strategy for the second player that is a best-response to p. **(5 points)**

5. You are given a Boolean formula ϕ with m clauses in conjunctive normal form, with at least 3 literals in each clause. For instance, ϕ could be

$$(X_2 \vee \overline{X_5} \vee \overline{X_8}) \wedge (X_1 \vee X_5 \vee \overline{X_6} \vee \overline{X_8}) \wedge (\overline{X_2} \vee X_6 \vee \overline{X_9}) \wedge (X_3 \vee X_4 \vee \overline{X_7} \vee \overline{X_{10}});$$

there are $m = 4$ clauses here, and the underlying Boolean variables are $X_1, \ldots, X_{10}$.

Given any such ϕ, show that there exists an assignment of truth values to the underlying Boolean variables that satisfies at least $7m/8$ of the given m clauses. **(5 points)**

6. Let $G = (V, E)$ be a graph with n vertices and minimum degree $\delta \geq 2$. Show that there is a partition of V into two subsets A and B such that $|A| \leq O(n \log(\delta+1)/(\delta+1))$, and such that each vertex of B has at least one neighbor in A and at least one neighbor in B. **(10 points)** (**Hint:** Start with a simple random process followed by a carefully-designed alteration.)