CMSC 858C, Spring 2009: Homework 3, due on April 30th (Thursday) at the start of class.

Notes: Please work on this with your group-mate(s): each group needs to submit only one solution. Consulting other sources (including the Web) is not allowed. Write your solutions **neatly**; if you are able to make partial progress by making some additional assumptions, then state these assumptions clearly and submit your partial solution.

1. Suppose you are given an undirected graph $G = (V, E)$ where the maximum degree of any vertex is Δ. (Recall that the degree of a vertex is the number of edges incident upon it.) Let $d(u)$ be the degree of vertex u; given a partition of V into two sets A and B, let $d_A(u)$ and $d_B(u)$ denote the number of neighbors of u in A and the number of neighbors of u in B, respectively. Show that there is a constant $C > 0$ and a partition of V into two sets A and B such that for all u,

$$d_A(u) \leq d(u)/2 + C\sqrt{\Delta \log \Delta} \text{ and } d_B(u) \leq d(u)/2 + C\sqrt{\Delta \log \Delta}.$$

(10 points)

2. Show that there is a constant $a > 0$ such that the following holds. We have an arbitrary graph $G = (V, E)$ with maximum degree Δ. Each vertex v also has a list of colors L_v; we want to color each vertex v with some color from L_v, so that we get a proper coloring (i.e., no two adjacent vertices get the same color). Prove that this is possible if the following holds: there is a non-negative value $b_{v,c}$ for all vertices v and all colors $c \in L_v$, such that:

- $\forall v, \ \sum_{c \in L_v} b_{v,c} = 1$; and
- $\forall (u, v) \in E, \ \sum_{c \in L_u \cap L_v} b_{u,c} \cdot b_{v,c} \leq a/\Delta$.

Try to get as large a value for the constant a as you can; however, your score for this problem will not depend on what constant a you get. **(5 points)**

3. Show that there is an integer constant $a > 0$ such that the following holds. We have an arbitrary graph $G = (V, E)$ with maximum degree Δ. Show that we can give a color from $\{1, 2, \ldots, a\Delta\}$ to each *edge* of G, so that the following hold:

- no two edges that share an end-point get the same color;
- no *even-length* cycle has only two colors given to its set of edges.

(**Hint:** Do an appropriate random construction. In addition to certain other bad events, associate one bad event $A_{e,f}$ with every pair of edges e and f that share an end-point. Use the *asymmetric* version of the Local Lemma; thus, in particular, you need to come up with suitable values $x_{e,f}$. Take $x_{e,f}$ a constant times $\Pr[A_{e,f}]$, and come up with suitable choices for the other parameters you need to define for the LLL.) **(15 points)**

4. In this problem, we prove that the Janson inequality parameter Δ is at most $O(\log N)$ in the Garg-Konjevod-Ravi Group Steiner Tree algorithm for trees, as claimed in class. Recall from class that we have a tree T with root r and n nodes. There are k disjoint groups $S_1, S_2, \ldots, S_k$, all of which are sets of leaves; also, $\max_i\ |S_i| = N$. Specifically, we fix a group S_i, and want to show that Δ_i, the Janson inequality parameter for the set of leaves that correspond to S_i, is at most $\ln |S_i|$. As in class, we let x_f be the fractional value of edge f, and if j is a leaf, then let $\text{pe}(j)$ denote the unique ("parent edge") incident on j.

(a). Suppose $j, j' \in S_i$. We will say that $j \sim j'$ if and only if **(i)** $j \neq j'$ and **(ii)** the least common ancestor of j and j' in G is not the root r. If $j \sim j'$, let $lca(j, j')$ denote the least common ancestral edge of j and j' in T'. Show that

$$\Delta_i = \sum_{j,j' \in S_i:\ j \sim j', x_{lca(j,j')} > 0} \frac{x_{\text{pe}(j)} x_{\text{pe}(j')}}{x_{lca(j,j')}}.$$

(5 points)

(b). We will now prove the following key fact:

$$\text{If } x_{\text{pe}(j)} > 0, \text{ then } x_{\text{pe}(j)} \cdot \sum_{j' \in S_i:\ j \sim j'} \frac{x_{\text{pe}(j')}}{x_{lca(j,j')}} \leq x_{\text{pe}(j)} \ln(1/x_{\text{pe}(j)}). \qquad (1)$$

We now take some steps toward proving (1). Suppose $x_{\text{pe}(j)} = z \in (0, 1]$. We need some extra notation. Let $e_0, e_1, \ldots, e_t$ be the sequence of edges that we encounter as we walk up the tree starting from j; let $y_\ell = x_{e_\ell}$. Thus we have $z = y_0 \leq y_1 \leq y_2 \leq \cdots \leq y_t \leq 1$. Next, for $\ell = 0, 1, \ldots, \ell$, let $A_\ell = \sum_{j' \in (T(e_\ell) \cap S_i)} x_{\text{pe}(j')}$. Then, it is not hard to see that the left-hand side in the statement of (1) equals

$$z \cdot \sum_{\ell=1}^{t} \frac{A_\ell - A_{\ell-1}}{y_\ell}. \qquad (2)$$

The sum in (2) is clearly bounded by the maximum of the following optimization problem, whose variables are the y_ℓ and A_ℓ. (The optimization problem has a maximum since the domain is a polytope and since the objective function is continuous in the domain.)

$$OPT(z,t): \quad \text{maximize } \textstyle\sum_{\ell=1}^{t} \frac{A_\ell - A_{\ell-1}}{y_\ell} \text{ subject to}$$

$$\begin{aligned}
A_0 &= z; \\
y_0 &= z; \\
y_t &\leq 1; \\
A_\ell &\leq A_{\ell+1}, \quad \ell = 0, 1, \ldots, t-1; \\
y_\ell &\leq y_{\ell+1}, \quad \ell = 0, 1, \ldots, t-1; \\
A_\ell &\leq y_\ell, \quad \ell = 0, 1, \ldots, t.
\end{aligned} \qquad (3)$$

Constraint (3) holds since the following constraint (4) is a constraint in our LP relaxation:

$$\sum_{j \in (L(f) \cap S_i)} x_{\mathrm{pe}(j)} \le x_f \quad \text{for every edge } f \text{ and every group } S_i. \tag{4}$$

Fix any feasible solution $\{y_\ell, A_\ell : \ell \ge 0\}$ to the above optimization problem.

- If v is the objective function value of this solution to the optimization problem, show that

$$v \le 1 - z/y_1 + \sum_{\ell=1}^{t-1} (1 - y_\ell / y_{\ell+1}). \tag{5}$$

(5 points)

- Take any ℓ, $2 \le \ell \le t-1$. If we keep all variables but y_ℓ fixed, see when the r.h.s. of (5) is maximized. Start with this idea to show that

$$v \le 1 - z/y_1 + \ln(1/y_1). \tag{6}$$

(5 points)

- Use (6) to show that $v \le \ln(1/z)$. This will then prove (1). **(5 points)**

(c). Show, using (1), that $\Delta_i \le \ln |S_i|$. **(5 points)**

5. We have a set V of n elements, and m distinct subsets $S_1, S_2, \ldots, S_m$ of V, each having cardinality t. Our goal is to choose a subset W of V with "many" elements, subject to the constraint that no S_i (for $i = 1, 2, \ldots, m$) be a subset of W.

Consider the following algorithm $\mathcal{A}$ for this problem. Let V be the set $\{1, 2, \ldots, n\}$. Independently for each $i \in V$, choose a number X_i uniformly at random from the set $\{1, 2, \ldots, n^3\}$. Now define a set W as follows: for each $i \in V$, $i \in W$ iff there is no set S_j such that: (i) $i \in S_j$, and (ii) for all $k \in S_j$, $X_i \ge X_k$.

(a). Show that $\mathcal{A}$ always produces a feasible solution to our problem. **(5 points)**

(b). Suppose $i \in V$ lies in a_i of the sets $S_1, S_2, \ldots, S_m$. Show that the expected size of the set W produced by $\mathcal{A}$ is at least

$$(1/n^3) \cdot \sum_{i=1}^{n} \sum_{\ell=1}^{n^3} \left(1 - (\ell/n^3)^{t-1}\right)^{a_i}.$$

(10 points)

6. Suppose we generate a random graph G from the $G(2t, 1/2)$-model; i.e., we take $2t$ vertices, and put an edge between each pair of vertices independently, with probability 1/2. Prove that the probability that *all* vertices of G have degree at most t, is at least $1/4^t$. **(5 points)**