CMSC 132: Object-Oriented Programming II

Algorithmic Complexity I

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Algorithm Efficiency

- **Efficiency**
  - **Amount of resources used by algorithm**
    - **Time, space**
- **Measuring efficiency**
  - **Benchmarking**
  - **Asymptotic analysis**

Benchmarking

- **Approach**
  - **Pick some desired inputs**
  - **Actually run implementation of algorithm**
  - **Measure time & space needed**
- **Industry benchmarks**
  - **SPEC – CPU performance**
  - **MySQL – Database applications**
  - **WinStone – Windows PC applications**
  - **MediaBench – Multimedia applications**
  - **Linpack – Numerical scientific applications**

Benchmarking

- **Advantages**
  - **Precise information for given configuration**
    - **Implementation, hardware, inputs**
- **Disadvantages**
  - **Affected by configuration**
    - **Data sets (often too small)**
      - **a dataset that was the right size 3 years ago is likely too small now**
    - **Hardware**
    - **Software**
  - **Affected by special cases (biased inputs)**
  - **Does not measure intrinsic efficiency**

Asymptotic Analysis

- **Approach**
  - **Mathematically analyze efficiency**
  - **Calculate time as function of input size n**
    - **T ≈ O(f(n))**
    - **T is on the order of f(n)**
    - **“Big O” notation**
- **Advantages**
  - **Measures intrinsic efficiency**
  - **Dominates efficiency for large input sizes**
  - **Programming language, compiler, processor irrelevant**

Search Example

- Number guessing game
  - Pick a number between 1…n
  - Guess a number
  - Answer “correct”, “too high”, “too low”
  - Repeat guesses until correct number guessed

Linear Search Algorithm

- Algorithm
  - Guess number = 1
  - If incorrect, increment guess by 1
  - Repeat until correct
- Example
  - Given number between 1…100
  - Pick 20
  - Guess sequence = 1, 2, 3, 4 … 20
  - Required 20 guesses

Linear Search Algorithm

- **Analysis of # of guesses needed for 1…n**
  - **If number = 1, requires 1 guess**
  - **If number = n, requires n guesses**
  - **On average, needs n/2 guesses**
  - **Time = O(n) = Linear time**

Binary Search Algorithm

- **Algorithm**
  - **Set low and high to be lowest and highest possible value**
  - **Guess middle = (low+high)/2**
  - **If too large, set high = middle-1**
  - **If too small, set low = middle+1**
  - **Repeat until guess correct**

Binary Search Algorithm

- **Example**
  - **Given number between 1…100**
  - **secret number we are trying is find is 20**
  - **Guesses**
    - **low = 1, high = 100, guess 50, Answer = too large**
    - **low = 1, high = 49, guess 25, Answer = too large**
    - **low = 1, high = 24, guess 12, Answer = too small**
    - **low = 13, high = 24, guess 18, Answer = too small**
    - **low = 19, high = 24, guess 21, Answer = too large**
    - **low = 19, high = 20, guess 19, Answer = too small**
    - **low = 20, high = 20, guess 20, Answer = correct**
    - **Required 7 guesses**

Binary Search Algorithm

- Analysis of # of guesses needed for 1…n
  - If number = n/2, requires 1 guess
  - If number = 1, requires $\log_2(n)$ guesses
  - If number = n, requires $\log_2(n)$ guesses
  - On average, needs $\log_2(n)$ guesses
  - Time = $O(\log_2(n)) = O(\log(n))$ = Log time

Search Comparison

- For number between 1…100
  - Simple algorithm = 50 steps
  - Binary search algorithm = $\log_2(n)$ = 7 steps
- For number between 1…100,000
  - Simple algorithm = 50,000 steps
  - Binary search algorithm = $\log_2(n)$ (about 17 steps)
- Binary search is much more efficient!

Asymptotic Complexity

- **Comparing two linear functions**

Size	Running Time	
	n/2	4n+3
64	32	259
128	64	515
256	128	1027
512	256	2051

Asymptotic Complexity

- Comparing two functions
  - n/2 and 4n+3 behave similarly
  - Run time roughly doubles as input size doubles
  - Run time increases linearly with input size
- For large values of n
  - Time(2n) / Time(n) approaches exactly 2
- Both are O(n) programs

Asymptotic Complexity

- **Comparing two log functions**

Size	Running Time	
	$\log_2(n)$	$5 * \log_2(n) + 3$
64	6	33
128	7	38
256	8	43
512	9	48

Asymptotic Complexity

- Comparing two functions
  - $\log_2(n)$ and $5 * \log_2(n) + 3$ behave similarly
  - Run time roughly increases by constant as input size doubles
  - Run time increases logarithmically with input size
- For large values of n
  - Time(2n) – Time(n) approaches constant
  - Base of logarithm does not matter
    - Simply a multiplicative factor

$$\log_a N = (\log_b N) / (\log_b a)$$

- Both are O(log(n)) programs

Asymptotic Complexity

- **Comparing two quadratic functions**

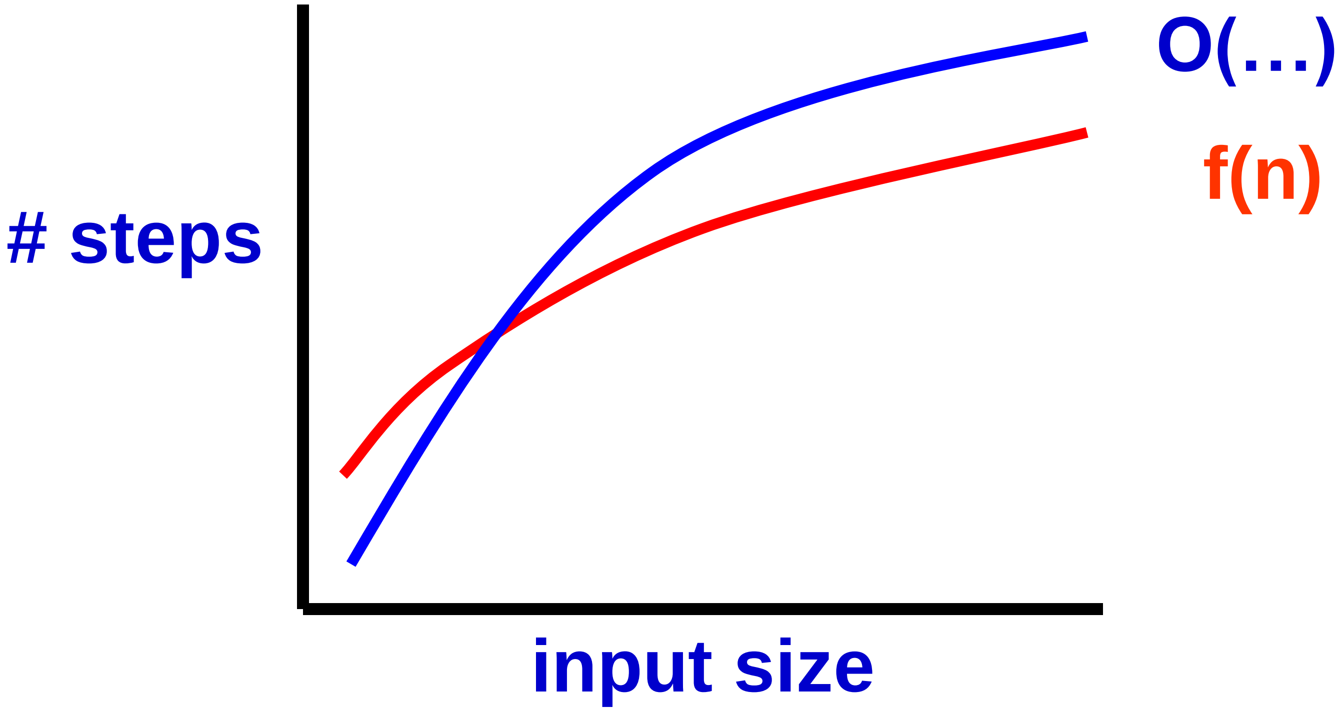
Size	Running Time	
	n^2	$2\ n^2 + 8$
2	4	16
4	16	40
8	64	132
16	256	520

Asymptotic Complexity

- Comparing two functions
  - n^2 and $2\ n^2 + 8$ behave similarly
  - Run time roughly increases by 4 as input size doubles
  - Run time increases quadratically with input size
- For large values of n
  - Time(2n) / Time(n) approaches 4
- Both are O(n^2) programs

Big-O Notation

- Represents
  - Upper bound on number of steps in algorithm
    - For sufficiently large input size
  - Intrinsic efficiency of algorithm for large inputs



Formal Definition of Big-O

- Function f(n) is O(g(n)) if
  - For some positive constants M, N_0
  - $M \times g(n) \geq f(n)$, for all $n \geq N_0$
- Intuitively
  - For some coefficient M & all data sizes $\geq N_0$
    - $M \times g(n)$ is always greater than f(n)

Big-O Examples

- $5n + 1000 \Rightarrow O(n)$
  - Select $M = 6$, $N_0 = 1000$
  - For $n \geq 1000$
    - $6n \geq 5n+1000$ is always true
  - Example $\Rightarrow$ for $n = 1000$
    - $6000 \geq 5000 + 1000$

Big-O Examples

- $2n^2 + 10n + 1000 \Rightarrow O(n^2)$
  - Select $M = 4$, $N_0 = 100$
  - For $n \geq 100$
    - $4n^2 \geq 2n^2 + 10n + 1000$ is always true
  - Example $\Rightarrow$ for $n = 100$
    - $40000 \geq 20000 + 1000 + 1000$

Observations

- **For large values of n**
  - **Any O(log(n)) algorithm is faster than O(n)**
  - **Any O(n) algorithm is faster than O(n^2)**
- **Asymptotic complexity is fundamental measure of efficiency**

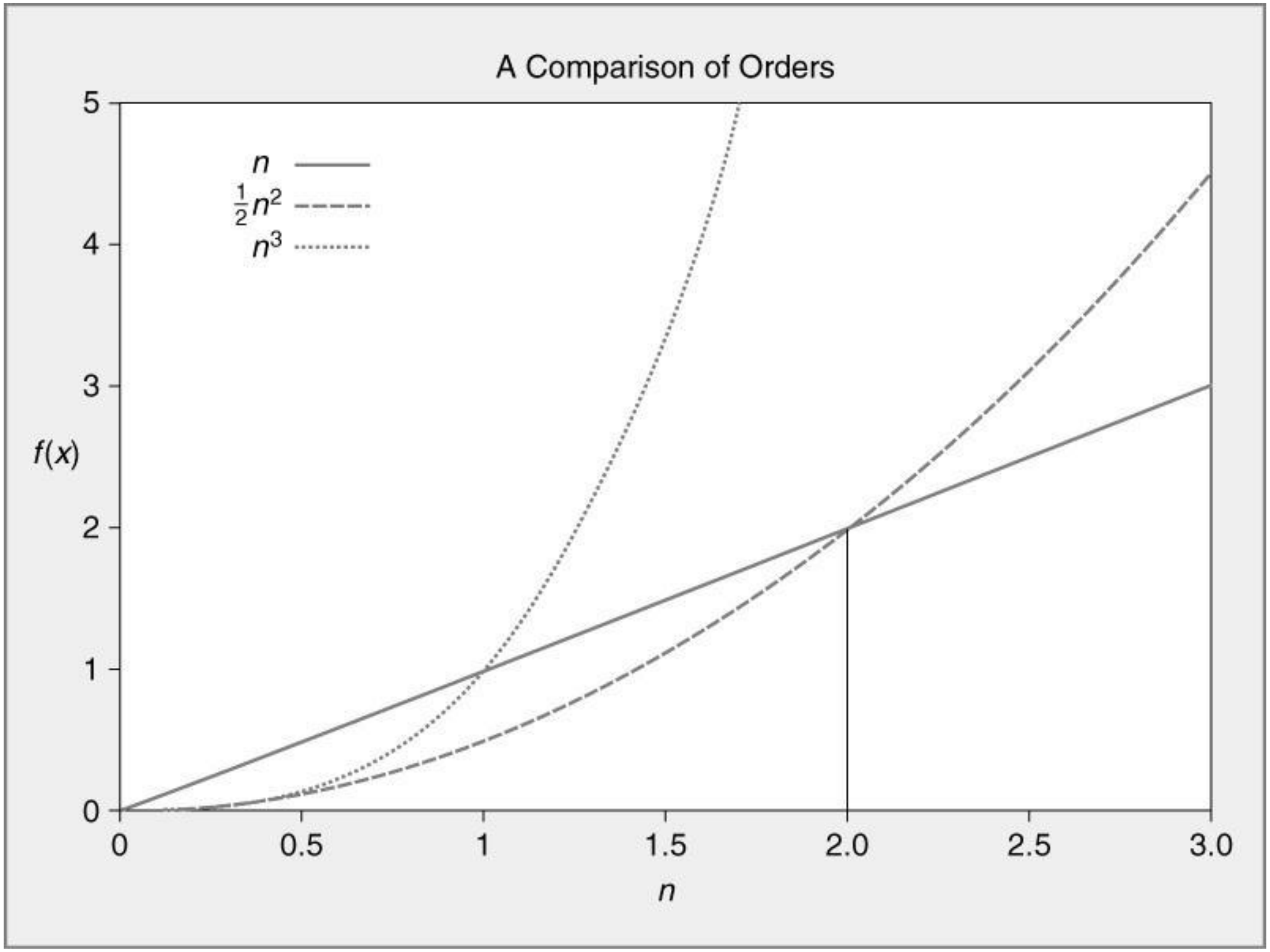
Asymptotic Complexity Categories

Complexity	Name	Example
$O(1)$	Constant	Array access
$O(\log(n))$	Logarithmic	Binary search
$O(n)$	Linear	Largest element
$O(n \log(n))$	N log N	Optimal sort
$O(n^2)$	Quadratic	2D Matrix addition
$O(n^3)$	Cubic	2D Matrix multiply
$O(n^k)$	Polynomial	Linear programming
$O(k^n)$	Exponential	Integer programming
$O(n!)$	Factorial	Brute-force search TSP
$O(n^n)$	N to the N	

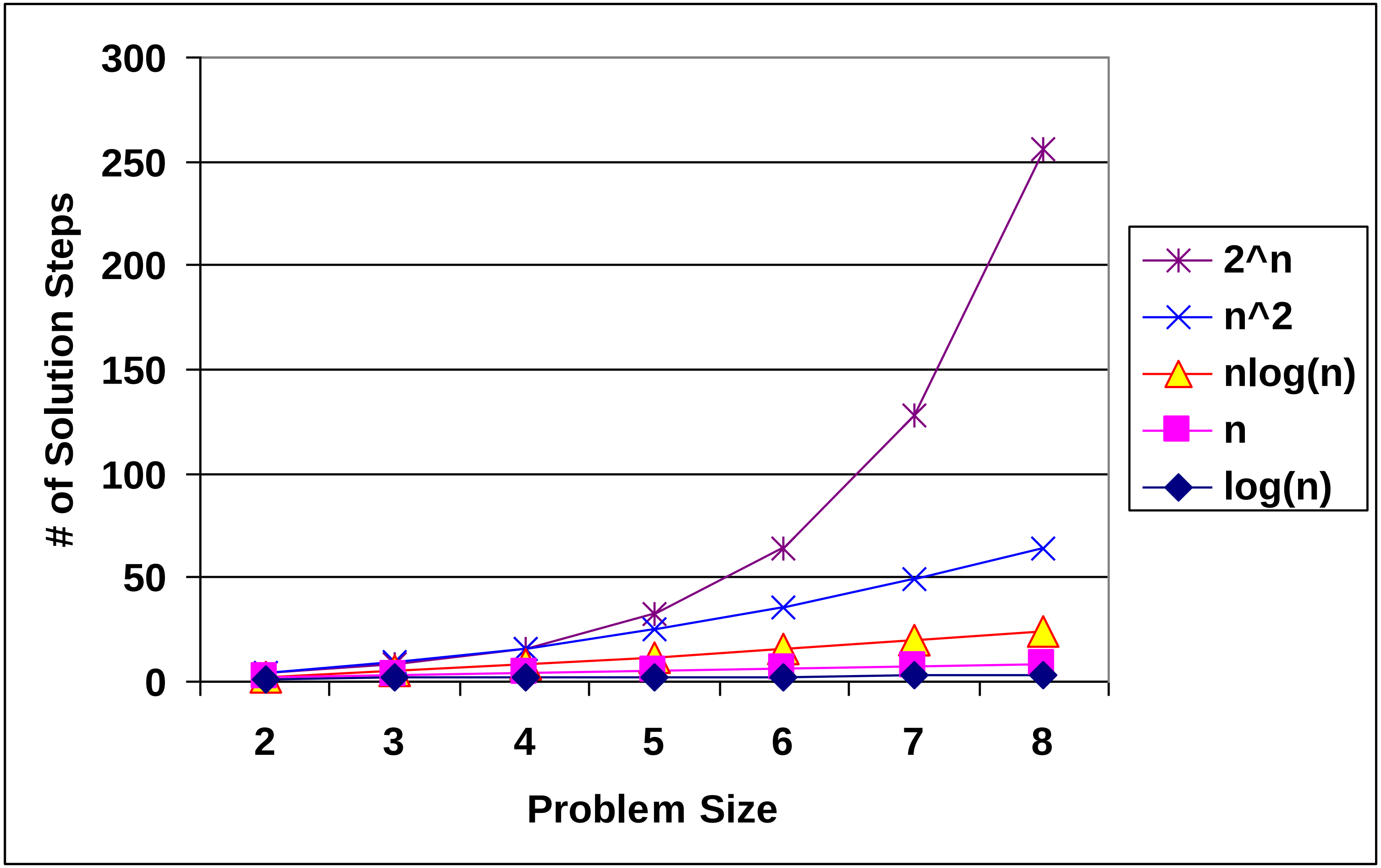
From smallest to largest

For size n, constant $k > 1$

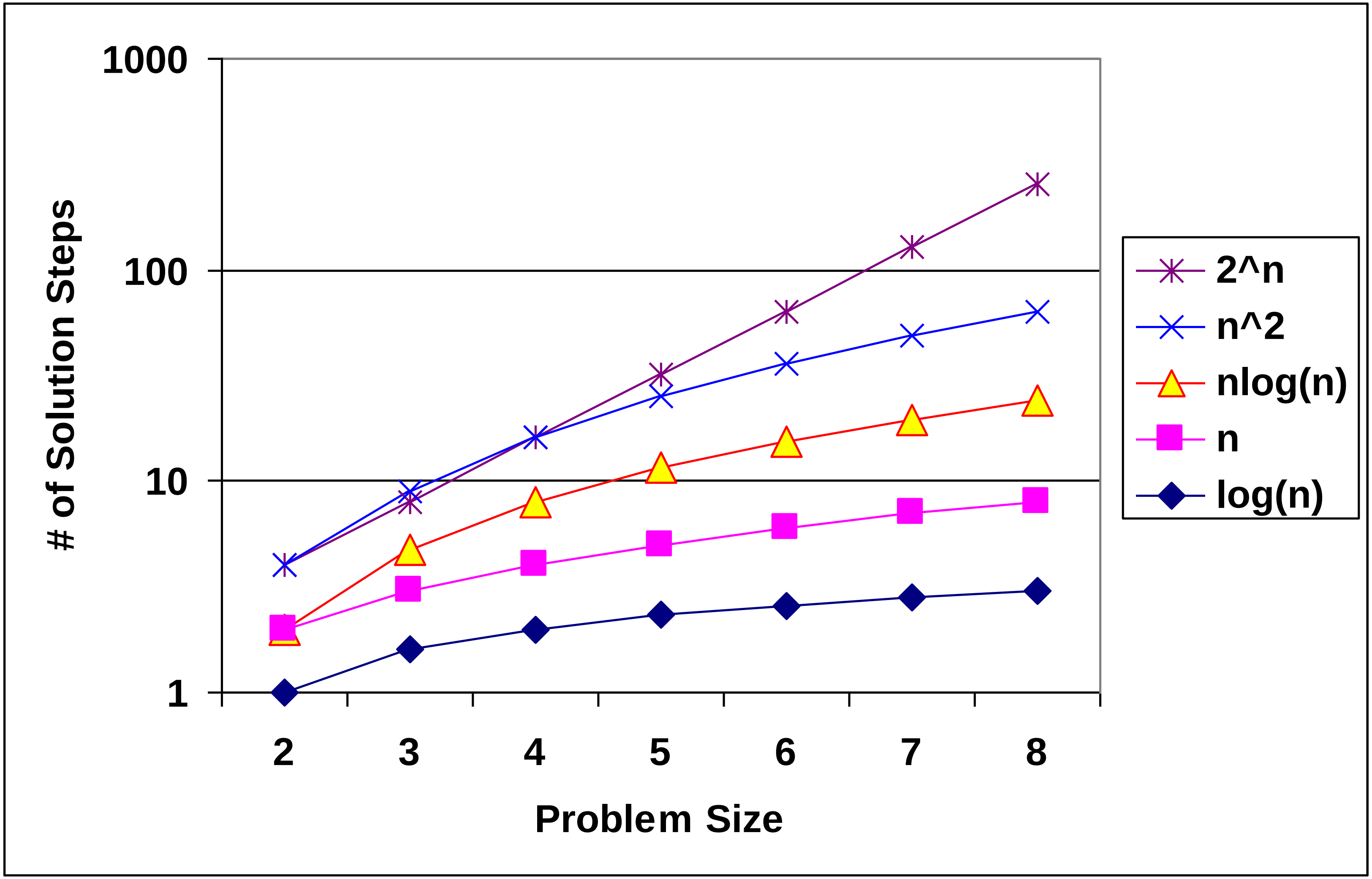
Comparison of Complexity



Complexity Category Example



Complexity Category Example

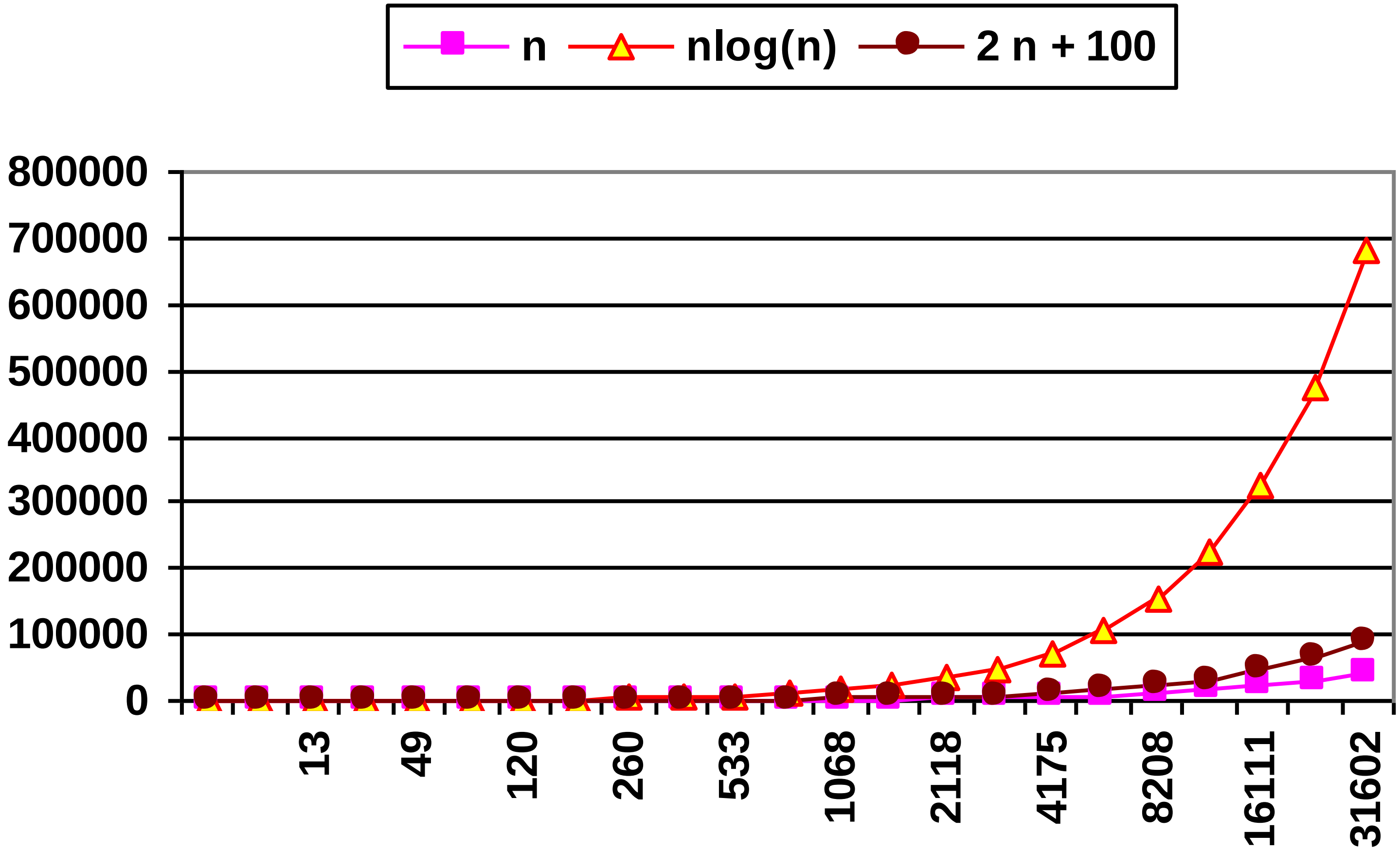


Calculating Asymptotic Complexity

- As n increases
  - Highest complexity term dominates
  - Can ignore lower complexity terms
- Examples
  - $2\ n + 100 \Rightarrow O(n)$
  - $n \log(n) + 10\ n \Rightarrow O(n\log(n))$
  - $\frac{1}{2}\ n^2 + 100\ n \Rightarrow O(n^2)$
  - $n^3 + 100\ n^2 \Rightarrow O(n^3)$
  - $1/100\ 2^n + 100\ n^4 \Rightarrow O(2^n)$

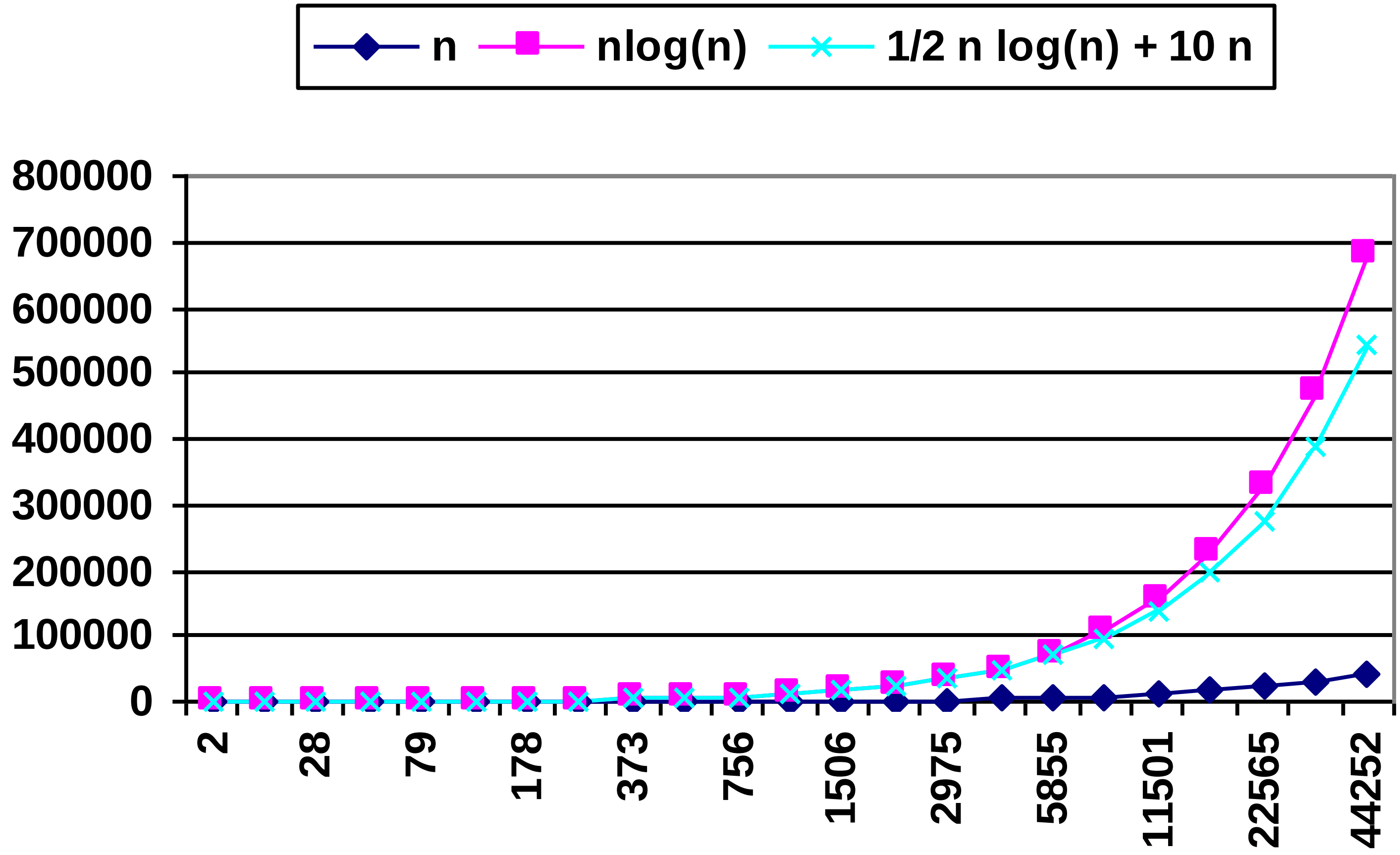
Complexity Examples

- $2n + 100 \Rightarrow O(n)$



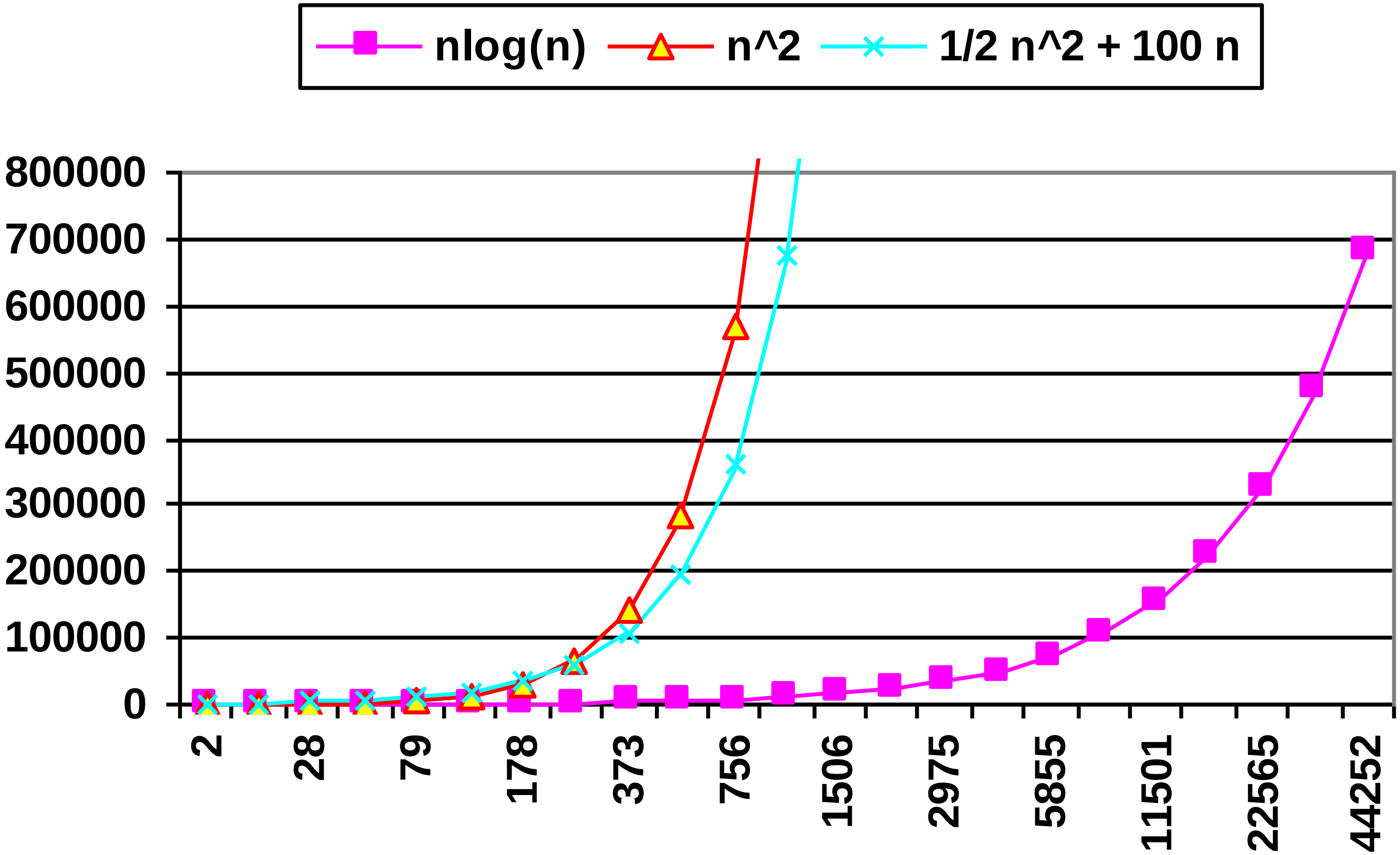
Complexity Examples

- $\frac{1}{2}$ n log(n) + 10 n $\Rightarrow$ O(nlog(n))



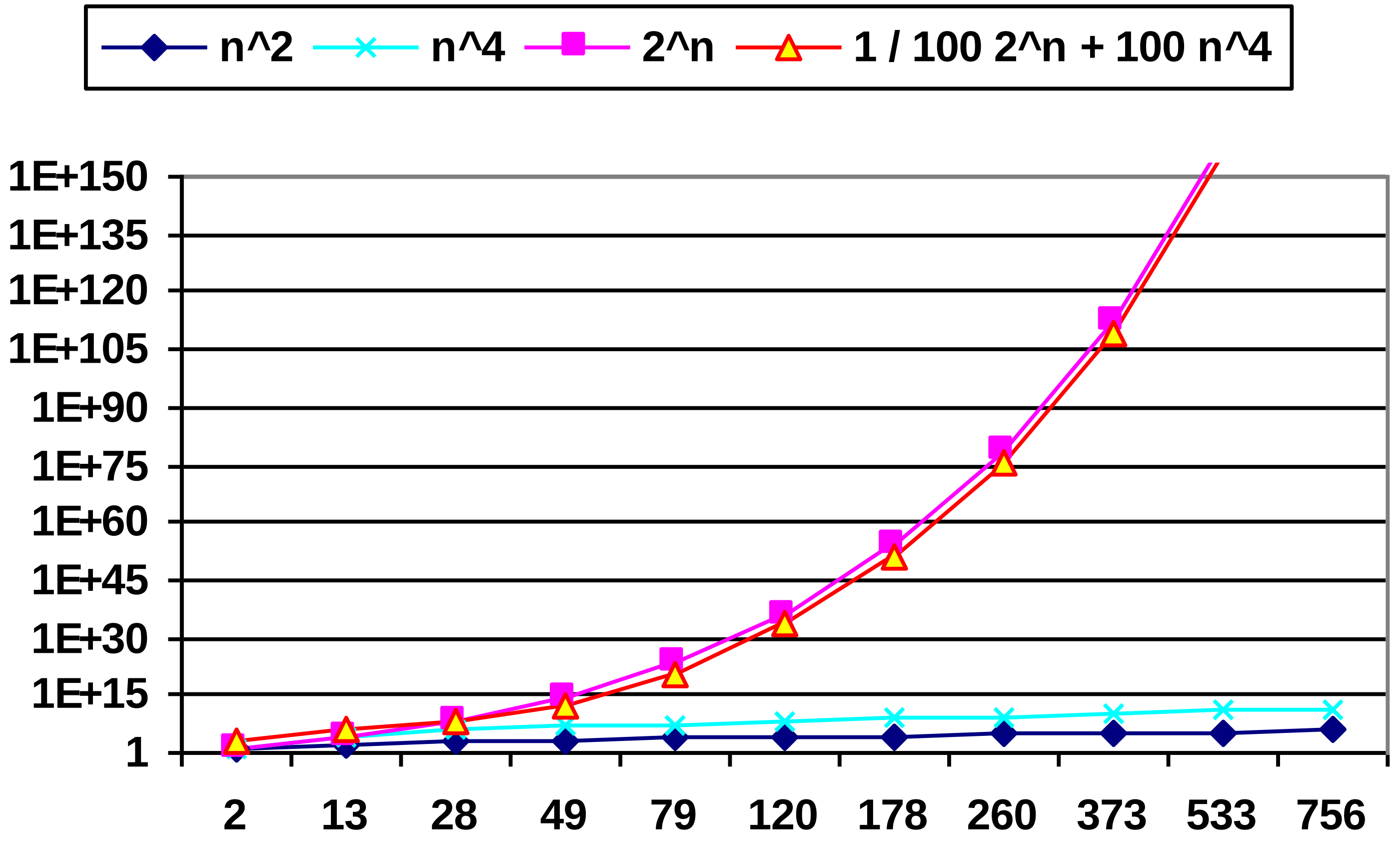
Complexity Examples

- $\frac{1}{2} n^2 + 100\, n \Rightarrow O(n^2)$



Complexity Examples

- **$1/100\ 2^n + 100\ n^4 \Rightarrow O(2^n)$**



Types of Case Analysis

- **Can analyze different types (cases) of algorithm behavior**
- **Types of analysis**
  - **Best case**
  - **Worst case**
  - **Average case**
  - **Amortized**

Types of Case Analysis

- **Best case**
  - Smallest number of steps required
  - Not very useful
  - Example ⇒ Find item in first place checked

Types of Case Analysis

- **Worst case**
  - **Largest number of steps required**
  - **Useful for upper bound on worst performance**
    - **Real-time applications (e.g., multimedia)**
    - **Quality of service guarantee**
  - **Example $\Rightarrow$ Find item in last place checked**

Quicksort Example

- **Quicksort**
  - **One of the fastest comparison sorts**
  - **Frequently used in practice**
- **Quicksort algorithm**
  - **Pick pivot value from list**
  - **Partition list into values smaller & bigger than pivot**
  - **Recursively sort both lists**

Quicksort Example

- **Quicksort properties**
  - **Average case = O(nlog(n))**
  - **Worst case = O(n^2)**
    - **Pivot ≈ smallest / largest value in list**
    - **Picking from front of nearly sorted list**
- **Can avoid worst-case behavior**
  - **Select random pivot value**

Types of Case Analysis

- **Average case**
  - **Number of steps required for “typical” case**
  - **Most useful metric in practice**
  - **Different approaches**
    - **Average case**
    - **Expected case**

Approaches to Average Case

- **Average case**
  - **Average over all possible inputs**
    - **Assumes all inputs have the same probability**
  - **Example**
    - **Case 1 = 10 steps, Case 2 = 20 steps**
    - **Average = 15 steps**
- **Expected case**
  - **Weighted average over all possible inputs**
    - **Based on probability of each input**
  - **Example**
    - **Case 1 (90%) = 10 steps, Case 2 (10%) = 20 steps**
    - **Average = 11 steps**

Average Case Example

- Example problem
  - Average # of comparisons needed to find a number in the (sorted) array A[ ] = {1, 4, 8, 12, 15} using
    - Linear search
      - Start from beginning, compare elements one at a time
    - Binary search
      - Start from middle of array at index k, compare element
      - If not element, repeat for top or bottom half of remaining array depending on whether element is smaller or greater than A[k]

Average Case : Linear Search

- **Algorithm**
  - **Find # of comparisons needed for each case**
    - **1 $\rightarrow$ 1 comparison (1)**
    - **4 $\rightarrow$ 2 comparisons (1, 4)**
    - **8 $\rightarrow$ 3 comparisons (1, 4, 8)**
    - **12 $\rightarrow$ 4 comparisons (1, 4, 8, 12)**
    - **15 $\rightarrow$ 5 comparisons (1, 4, 8, 12, 15)**
  - **Calc average = total # of comparisons / # cases**
    - **Total # comparisons = 1 + 2 + 3 + 4 + 5 = 15**
    - **# cases = 5**
    - **Average = 3 comparisons / number**

Average Case : Binary Search

- Algorithm
  - Find # of comparisons needed for each case
    - 1 $\rightarrow$ 3 comparisons (8, 4, 1)
    - 4 $\rightarrow$ 2 comparisons (8, 4)
    - 8 $\rightarrow$ 1 comparisons (8)
    - 12 $\rightarrow$ 2 comparisons (8, 12)
    - 15 $\rightarrow$ 3 comparisons (8, 12, 15)
  - Calc average = total # of comparisons / # cases
    - Total # comparisons = 3 + 2 + 1 + 2 + 3 = 11
    - # cases = 5
    - Average = 2.2 comparisons / number

Average Case Example

- **Example problem 2**
  - **Average # of comparisons needed to find a number in a sorted array A[n] of size n using**
    - **Linear search**
    - **Binary search**
  - **For simplicity, we assume elements are stored in A[1] … A[n]**

Average Case : Linear Search

- **Algorithm**
  - Find # of comparisons needed for each case
    - A[1] → 1 comparison (A[1])
    - A[2] → 2 comparisons (A[1], A[2])
    - …
    - A[n] → n comparisons (A[1] … A[n])
  - Calc average = total # of comparisons / # cases
    - Total # comparisons = 1 + 2 + … + n = ½ n2 + 1
    - # cases = n
    - Average ≈ ½ n comparisons / number

Average Case : Binary Search

- Algorithm
  - Find # of comparisons needed for each case
    - A[n/2] → 1 comp (A[n/2])
    - A[n/4], A[3n/4] → 2 comps (A[n/2], A[n/4])
    - …
    - A[1], A[3]…A[n] → log2(n) comparisons
    - (A[n/2], A[n/4], A[n/8]…A[1])
  - Calc average = total # of comparisons / # cases
    - Total # comparisons = n/2 * log2(n) +
    - n/4 * log2(n)–1 + … + 1 = n log2(n)
    - # cases = n
    - Average ≈ log2(n) comparisons / number

Sample problem

- Given an array a of integers
  - find the subrange that has the maximum sum
    - e.g., find low, high that maximizes
      a[low] + a[low+1] + ... + a[high]
    - only non empty ranges (low <= high)
    - If a contained only nonnegative integers, would be low = 0, high = a.length -1
    - but a can contain negative numbers
    - Can assume that arithmetetic overflow isn't an issue

One solution

```
public static int findBestRange(int [] a) {
int bestSum = a[0];
for(int low = 0; low < a.length; low++)
  for(int high = low; high < a.length; high++) {
    int sum = 0;
    for(int i = low; i <= high; i++) sum += a[i];
    if (bestSum < sum)
      bestSum = sum;
     }
return bestSum;
}

// What is the complexity of the algorithm used here?
```

Can you find a better algorithm?