CMSC 132: Object-Oriented Programming II

Algorithmic Complexity II

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Overview

- Critical sections
- Comparing complexity
- Types of complexity analysis

Analyzing Algorithms

- **Goal**
  - **Find asymptotic complexity of algorithm**
- **Approach**
  - **Ignore less frequently executed parts of algorithm**
  - **Find critical section of algorithm**
  - **Determine how many times critical section is executed as function of problem size**

Critical Section of Algorithm

- **Heart of algorithm**
- **Dominates overall execution time**
- **Characteristics**
  - **Operation central to functioning of program**
  - **Contained inside deeply nested loops**
  - **Executed as often as any other part of algorithm**
- **Sources**
  - **Loops**
  - **Recursion**

Critical Section Example 1

- Code (for input size n)
  1. A
  2. for (int i = 0; i < n; i++)
  3.   B
  4. C

critical section

- Code execution
  - A $\Rightarrow$ once
  - B $\Rightarrow$ n times
  - C $\Rightarrow$ once
- Time $\Rightarrow 1 + n + 1 = O(n)$

Critical Section Example 2

- **Code (for input size n)**
  1. **A**
  2. **for (int i = 0; i < n; i++)**
  3. **B**
  4. **for (int j = 0; j < n; j++)**
  5. **C**
  6. **D**

critical section

- **Code execution**
  - **A $\Rightarrow$ once**
  - **C $\Rightarrow$ n^2 times**
  - **B $\Rightarrow$ n times**
  - **D $\Rightarrow$ once**
- **Time $\Rightarrow$ $1 + n + n^2 + 1 = O(n^2)$**

Critical Section Example 3

- **Code (for input size n)**
  1. **A**
  2. **for (int i = 0; i < n; i++)**
  3. **for (int j = i+1; j < n; j++)**
  4. **B**

critical section

- **Code execution**
  - **A ⇒ once**
  - **B ⇒ ½ n (n-1) times**
- **Time ⇒ 1 + ½ n^2 = O(n^2)**

Critical Section Example 4

- **Code (for input size n)**
  1. **A**
  2. **for (int i = 0; i < n; i++)**
  3. **for (int j = 0; j < 10000; j++)**
  4. **B**

critical section

- **Code execution**
  - **A ⇒ once**
  - **B ⇒ 10000 n times**
- **Time ⇒ 1 + 10000 n = O(n)**

Critical Section Example 5

- **Code (for input size n)**

```
for (int i = 0; i < n; i++)
   for (int j = 0; j < n; j++)
      A
for (int i = 0; i < n; i++)
   for (int j = 0; j < n; j++)
      B
```

critical sections

- **Code execution**
  - **A $\Rightarrow n^2$ times**
  - **B $\Rightarrow n^2$ times**
- **Time $\Rightarrow n^2 + n^2 = O(n^2)$**

Critical Section Example 6

- **Code (for input size n)**
  1. **i = 1**
  2. **while (i < n) {**
  3. **A**
  4. **i = 2 × i**

     }
  5. **B**

critical section

- **Code execution**
  - **A ⇒ log(n) times**
  - **B ⇒ 1 times**
- **Time ⇒ log(n) + 1 = O(log(n))**

Critical Section Example 7

- Code (for input size n)
  1. DoWork (int n)
  2.   if (n == 1)
  3.   A
  4.   else
  5.   DoWork(n/2)
  6.   DoWork(n/2)

critical sections

- Code execution
  - A $\Rightarrow$ 1 times
  - DoWork(n/2) $\Rightarrow$ 2 times
- Time(1) $\Rightarrow$ 1   Time(n) = 2 $\times$ Time(n/2) + 1

Recursive Algorithms

- Definition
  - An algorithm that calls itself
- Components of a recursive algorithm

1. Base cases
   - Computation with no recursion
2. Recursive cases
   - Recursive calls
   - Combining recursive results

Recursive Algorithm Example

- **Code (for input size n)**

```
DoWork (int n)
    if (n == 1)
        A
    else
        DoWork(n/2)
        DoWork(n/2)
```

base case

recursive cases

Comparing Complexity

- **Compare two algorithms**
  - **f(n), g(n)**
- **Determine which increases at faster rate**
  - **As problem size n increases**
- **Can compare ratio**

$$\lim_{n \to \infty} \frac{f(n)}{g(n)}$$

  - **If ∞, f() is larger**
  - **If 0, g() is larger**
  - **If constant, then same complexity**

Complexity Comparison Examples

- $\log(n)$ vs. $n^{½}$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \rightarrow \lim_{n\to\infty} \frac{\log(n)}{n^{½}} \rightarrow 0$$

- 1.001^{n} vs. n^{1000}

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \rightarrow \lim_{n\to\infty} \frac{1.001^{n}}{n^{1000}} \rightarrow ??$$

Not clear, use L'Hopital's Rule

Additional Complexity Measures

- Upper bound
  - Big-O $\Rightarrow O(\ldots)$
  - Represents upper bound on # steps
- Lower bound
  - Big-Omega $\Rightarrow \Omega(\ldots)$
  - Represents lower bound on # steps
- Combined bound
  - Big-Theta $\Rightarrow \Theta(\ldots)$
  - Represents combined upper/lower bound on # steps
  - Best possible asymptotic solution

2D Matrix Multiplication Example

- **Problem**
  - **C = A * B**

- **Lower bound**
  - **$\Omega(n^2)$** Required to examine 2D matrix
- **Upper bounds**
  - **$O(n^3)$** Basic algorithm
  - **$O(n^{2.807})$** Strassen's algorithm (1969)
  - **$O(n^{2.376})$** Coppersmith & Winograd (1987)
- **Improvements still possible (open problem)**
  - **Since upper & lower bounds do not match**

Additional Complexity Categories

Name	Description
P	Deterministic polynomial time
NP	Nondeterministic polynomial time
PSPACE	Polynomial space
EXPSPACE	Exponential space
Decidable	Can be solved by finite algorithm
Undecidable	Not solvable by finite algorithm

- If a problem has a algorithm that solves it in time X, then the problem is said to be in X
  - e.g., matrix multiplication is in P

Why do we care?

- **If a problem can't be solved in P time, then no algorithm can solve *all* cases exactly in a *reasonable* amount of time**
  - **But there might be an algorithm that always quickly gives close approximations**
  - **Or an algorithm that gives quick exact answers for the cases we care about**

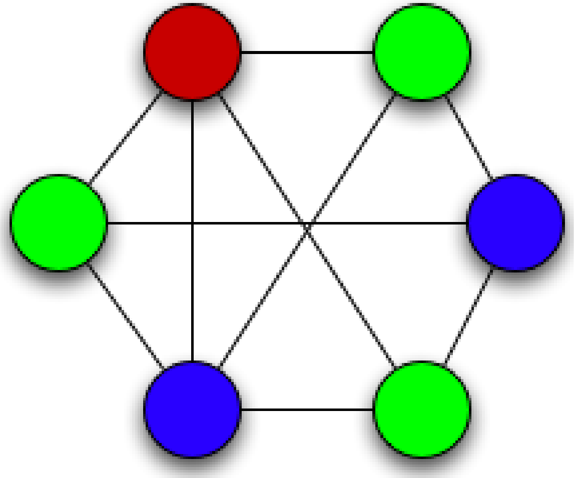
- **Issues such as PSPACE vs. EXPSPACE are interesting theoretical issues, and sometimes provide practice insight**

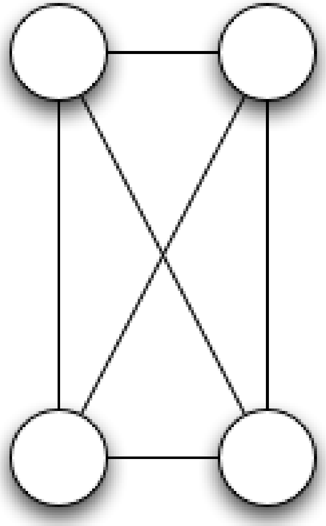
NP Time Algorithms

- **Two ways of thinking about it**
  - **First way:**
    - **Given a problem, and a potential answer, can we verify that the answer is correct in polynomial time?**
  - **Second way:**
    - **Whenever the algorithm wants, it can clone itself in two**
    - **If either clone finds an answer, the entire system produces an answer**

Graph 3-coloring

- Given a graph (vertices and undirected edges)
- Can you find a way to color each vertex either blue, red, or green
- Such that no two vertices connected by an edge have the same color?





Some graphs are hard

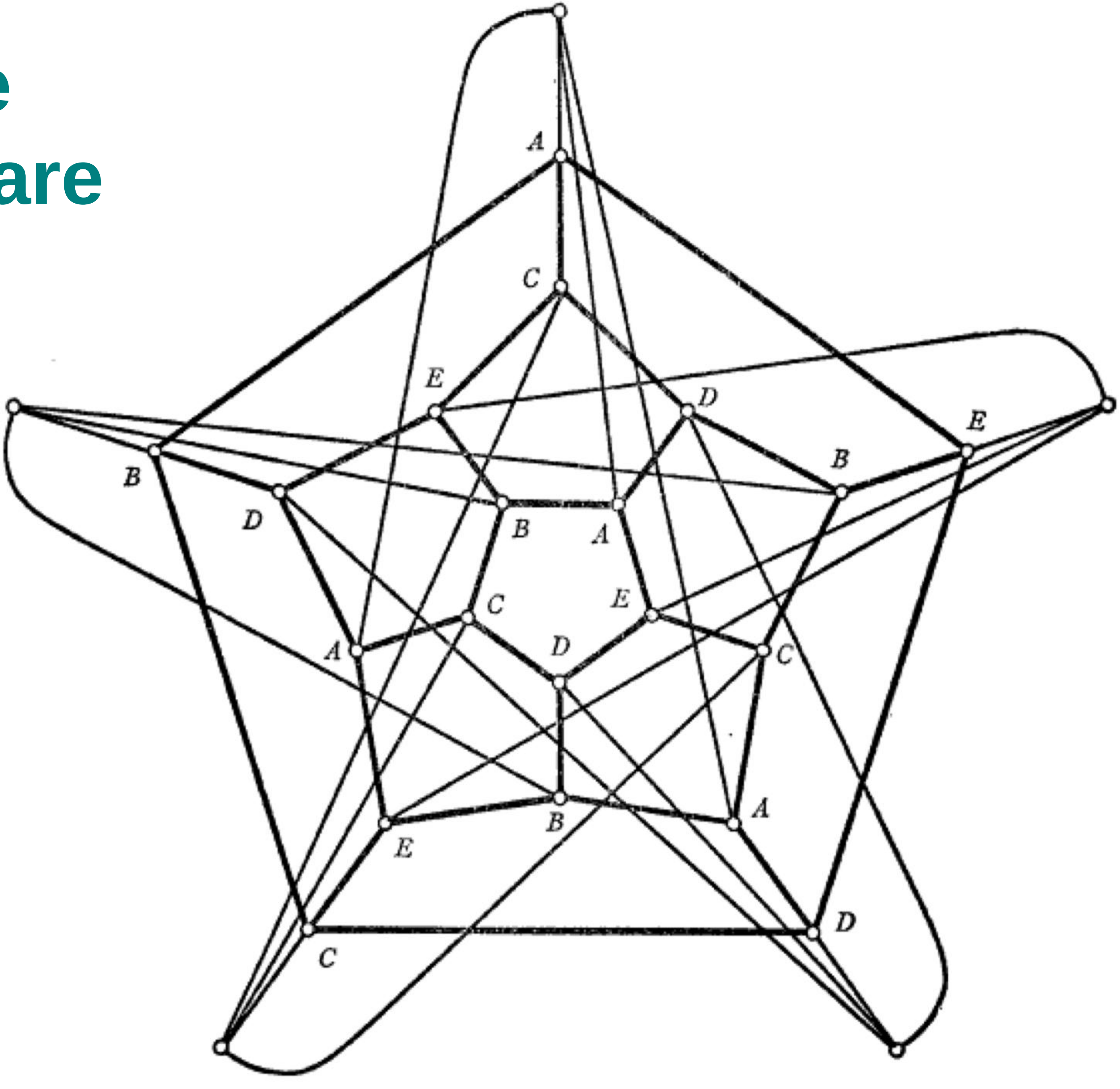


FIG. 1

3 coloring a graph is in NP

- **There exist NP algorithms to 3 color a graph**
  - **Guess a 3 coloring**
  - **Verify that the coloring is valid**
    - **Easy to do in O(n) time**

- **No one knows if there exists a polynomial time algorithm to find a 3 coloring for an arbitrary graph**
  - **If you come up with one, even one that runs in time $O(n^{100})$, you win one million dollars**
  - **Seriously**

NP Time Algorithm

- **Many interesting problems are solvable with an NP algorithm, but not known to be solvable with a P algorithm**
  - **Boolean satisfiability**
  - **Traveling salesman problem (TLP)**
  - **Bin packing**
- **Key to solving many optimization problems**
  - **Most efficient trip routes**
  - **Most efficient schedule for employees**
  - **Most efficient usage of resources**

NP Complete problems

- Some problems are NP-complete
  - They are in NP
  - And if a polynomial time algorithm existed for the program
  - Then every single last problem in NP could be solved in polynomial time
- Almost all problems that are in NP but are not known to be in P are NP-complete
  - But not all

P = NP?

- Are NP problems solvable in polynomial time?
  - Prove P=NP
    - Show polynomial time solution exists for any NP-complete problem
  - Prove P≠NP
    - Show no polynomial-time solution possible for some problem in NP
    - The expected answer
- The most important open problem in CS
  - $1 million prize offered by Clay Math Institute
  - Plus front page, NY Times, job offers galore, instant Ph.D. in Computer Science

Algorithmic Complexity Summary

- Asymptotic complexity
  - Fundamental measure of efficiency
  - Independent of implementation & computer platform
- Learned how to
  - Examine program
  - Find critical sections
  - Calculate complexity of algorithm
  - Compare complexity