Original notes by King-Ip Lin.

5 Binomial heaps

These are heaps that also provide the power to merge two heaps into one.

Definition 5.1 (Binomial tree) *A* binomial tree of height k *(denoted as B_k) is defined recursively as follows:*

1. *B_0 is a single node*
2. *B_{i+1} is formed by joining two B_i heaps, making one's root the child of the other*

Basic properties of B_k

- Number of nodes : 2^k
- Height : k
- Number of children of root (degree of root) : k

In order to store n nodes in binomial tress when $n \neq 2^k$, write n in binary notation; for every bit that is "1" in the representation, create a corresponding B_k tree, treating the rightmost bit as 0.
Example : $n = 13 = 1101_2 \longrightarrow$ use B_3, B_2, B_0

Definition 5.2 (Binomial heap) *A* binomial heap *is a (set of) binomial tree(s) where each node has a key and the heap-order property is preserved. We also have the requirement that for any given i there is at most one B_i.*

Algorithms for the binomial heap :

Find minimum Given n, there will be $\log n$ binomial trees and each tree's minimum will be its root. Thus only need to find the minimum among the roots.
$Time = \log n$

Insertion Invariant to be maintained : only 1 B_i tree for each i

Step 1: create a new B_0 tree for the new element

Step 2: $i \leftarrow 0$

Step 3: **while** there are still two B_i trees do
join the two B_i trees to form a B_{i+1} tree
$i \leftarrow i + 1$

Clearly this takes at most $O(\log n)$ time.

Deletion

Delete min Key observation: Removing the root from a B_i tree will form i binomial trees, from B_0 to B_{i-1}

Step 1: Find the minimum root (assume its B_k)

Step 2: Break B_k, forming k smaller trees

Step 3: **while** there are still at least two B_i trees do
join the two B_i trees to form a B_{i+1} tree
$i \leftarrow i+1$

Note that at each stage there will be at most three B_i trees, thus for each i only one join is required.

Delete

Step 1: Find the element

Step 2: Change the element key to $-\infty$

Step 3: Push the key up the tree to maintain the heap-order property

Step 4: Call Delete min

Steps 2 and 3 are grouped and called DECREASE_KEY(x)
Time for insertion and deletion : $O(\log n)$

5.1 Fibonacci Heaps (F-Heaps)

Amortized running time :

- Insert, Findmin, Union, Decrease-key : $O(1)$
- Delete-min, Delete : $O(\log n)$

Added features (compared to the binomial heaps)

- Individual trees are not necessary binomial (denote trees by B'_i)
- Always maintain a pointer to the smallest root
- permit many copies of B'_i

Algorithms for F-heaps:

Insert

Step 1: Create a new B'_0

Step 2: Compare with the current minimum and update pointer if necessary

Step 3: Store \$1 at the new root

(Notice that the \$ is only for accounting purposes, and the implementation of the data structure does not need to keep track of the \$'s.)

Delete-min

Step 1: Remove the minimum, breaking that tree into smaller tree again

Step 2: Find the new minimum, merge trees in the process, resulting in at most one B'_i tree for each i

Decrease-key

Step 1: Decrease the key

Step 2: **if** heap-order is violated
break the link between the node and its parent (note: resulting tree may not be a true binomial tree)

Step 3: Compare the new root with the current minimum and update pointer if necessary

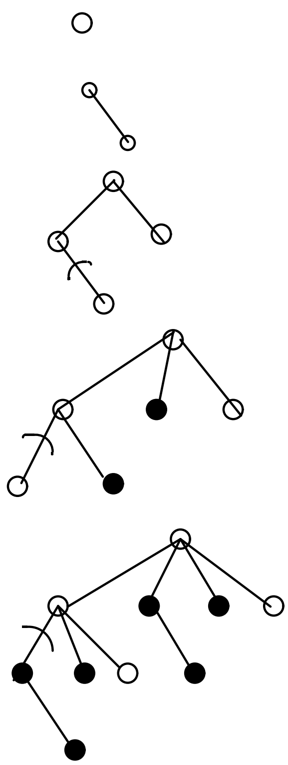
Step 4: Put \$1 to the new root

Step 5: Pay \$1 for the cut

Problem with the above algorithm: Can result in trees where the root has a very large number of children.
Solution:

- whenever a node is being cut
  - mark the parent of the cut node in the original tree
  - put \$2 on that node
- when a second child of that node is lost (by that time that node will have \$4), recursively cut that node from its parent, use the \$4 to pay for it:
  - \$1 for the cut
  - \$1 for new root
  - \$2 to its original parent
- repeat the recursion upward if necessary

Thus each cut requires only \$4.
Thus decrease-key takes amortized time $O(1)$
Define rank of the tree = Number of children of the root of the tree.
Consider the minimum number of nodes of a tree with a given rank:



	Rank	Worst case size	Size of binomial tree
B0	0	1	1
B1	1	2	2
B2	2	3	4
B3	3	5	8
B4	4	8	16

● Marked node from previous deletion

Figure 4: Minimum size B_i' trees

Original notes by Patchanee Ittarat.

6 F-heap

6.1 Properties

F-heaps have the following properties:

- maintain the minimum key in the heap all the time,
- relax the condition that we have at most one B_k tree for any k,i.e., we can have any number of B_k trees co-existing.
- trees are not true binomial trees.

For example,

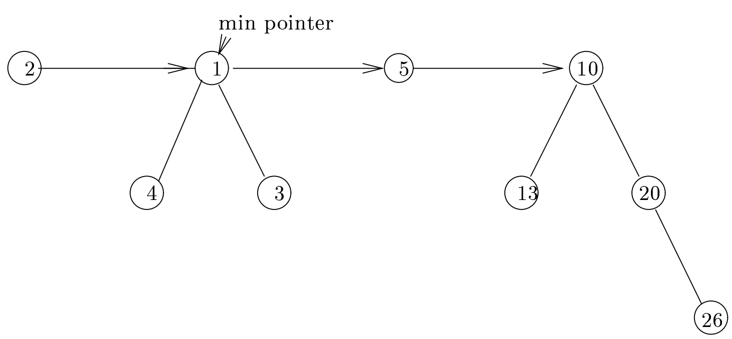


Figure 5: Example of F-heap

Property: size of a tree, rooted at x, is in exponential in the degree of x.

In Binomial trees we have the property that $size(x) \geq 2^k$, here we will not have as strong a property, but we will have the following:

$$size(x) \geq \phi^k$$

where k is degree of x and $\phi = \frac{1+\sqrt{5}}{2}$.

Note that since $\phi > 1$ this is sufficient to guarantee a $O(\log n)$ bound on the depth and the number of children of a node.

6.2 Decrease-Key operation

Mark strategy:

- when a node is cut off from its parent, tick one mark to its parent,
- when a node gets 2 marks, cut the edge to its parent and tick its parent as well,
- when a node becomes a root, erase all marks attached to that node,
- every time a node is cut, give \$1 to that node as a new root and \$2 to its parent.

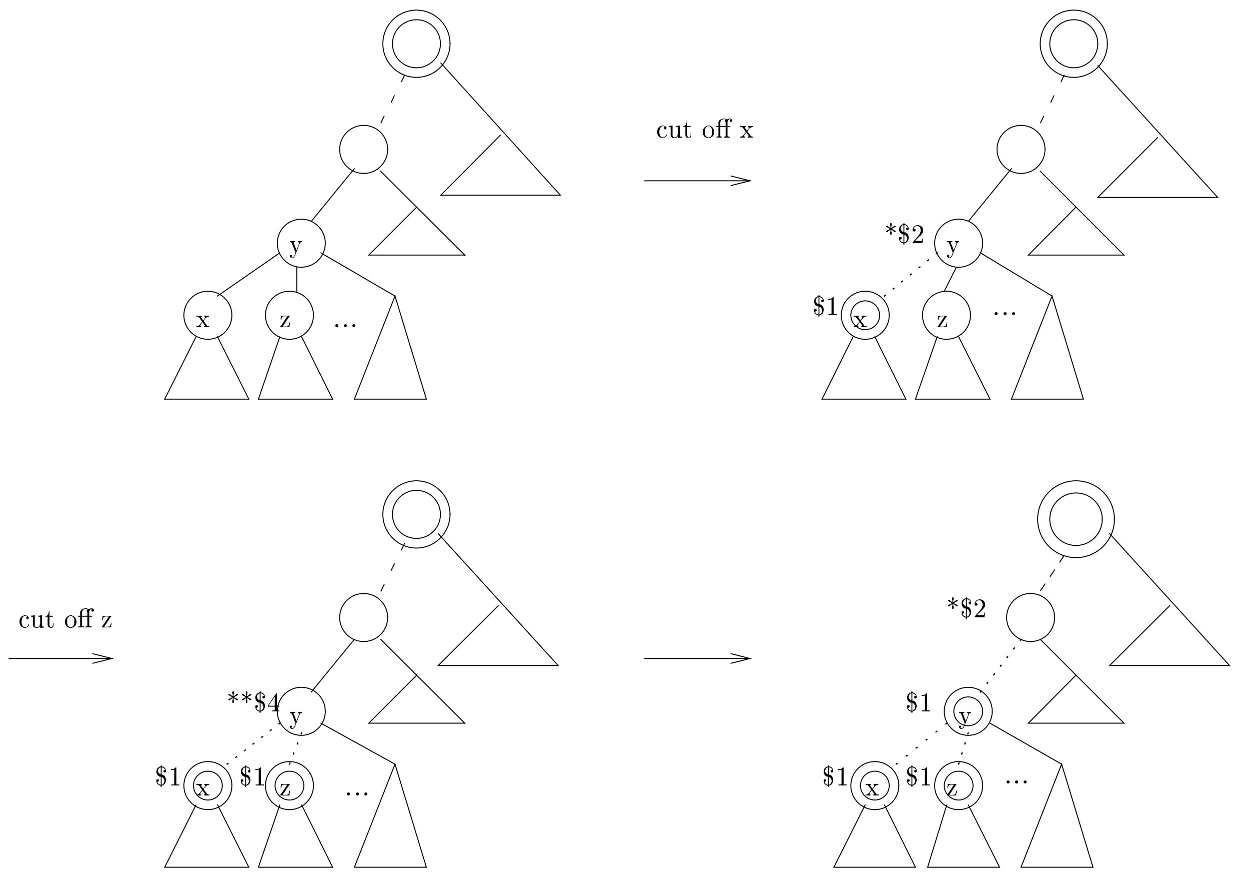


Figure 6: Marking Strategy

The cost of Decrease-Key operation = cost of cutting link + \$3. We can see that no extra dollars are needed when the parent is cut off recursively since when the cut off is done, that parent must have \$4 in hand (\$2 from each cut child and there must be 2 children have been cut), then we use those \$4 dollars to pay for cutting link cost (\$1), giving to its parent (\$2), and for itself as a new root (\$1).

Example: How does Decrease-Key work (see CLR also)?

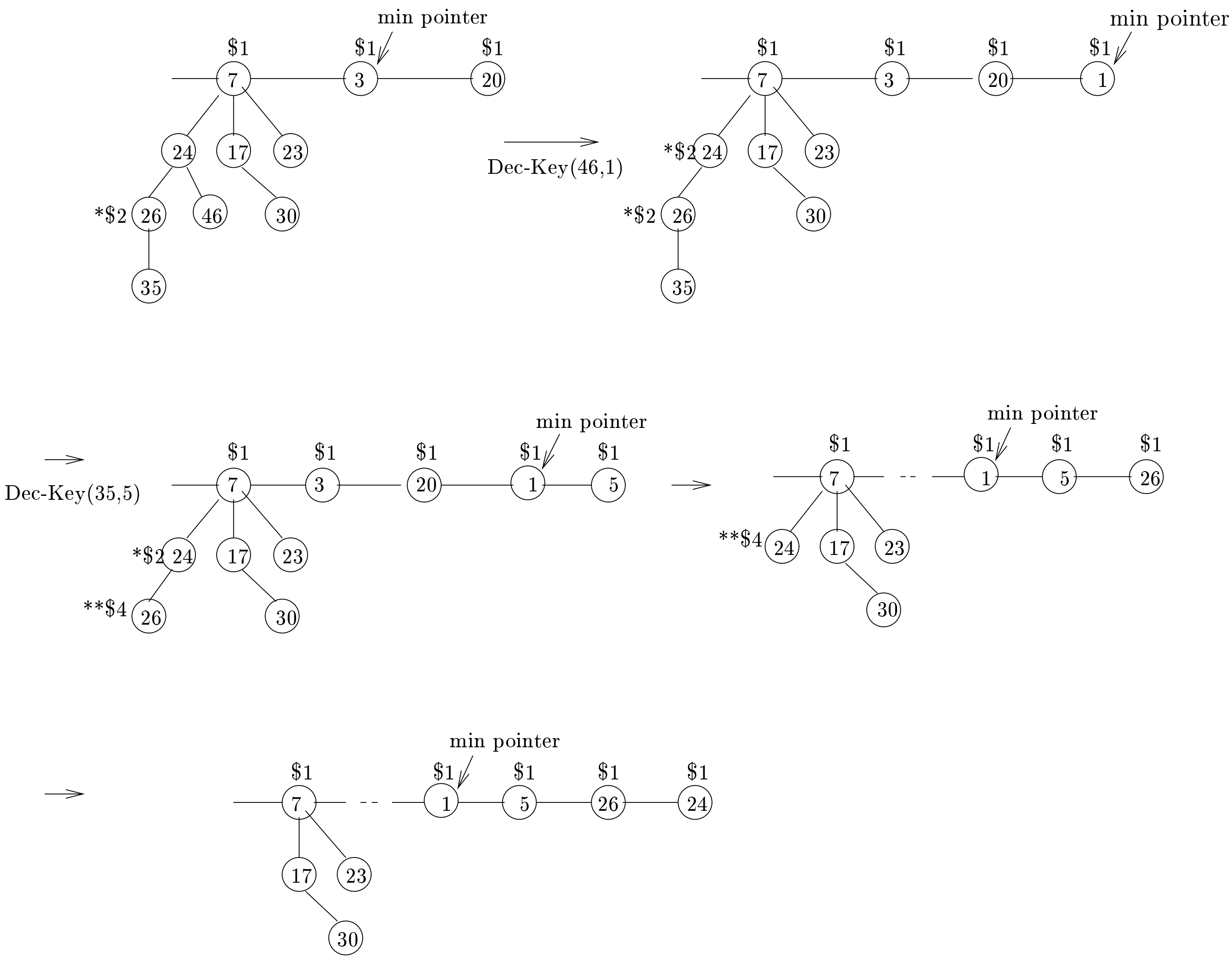


Figure 7: Example

The property we need when two trees are merged, is the degree of the roots of both trees should be the same.

Let y_1 be the oldest child and y_k be the youngest child. Consider y_i, at the point y_i was made a child of the root x, x had degree at least $i-1$ and so does y_i. Since y_i has lost at most 1 child, now y_i has at least degree $i-2$.

We now study some properties about Fibonacci numbers, and see how they relate to the sizes of the trees with the decrease-key operation. Define

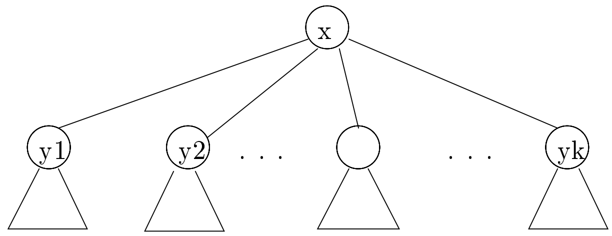


Figure 8: Example

$$F_k = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

These numbers are called Fibonacci numbers.

Property 6.1 $F_{k+2} = 1 + \sum_{i=1}^{k} F_i$

Proof:
[By induction]

$$\begin{aligned} k = 1: \qquad F_3 &= 1 + F1 \\ &= 1 + 1 \\ &= F_1 + F_2 \quad (\text{where } F_2 = F_1 + F_0 = 1 + 0) \end{aligned}$$

Assume $F_{k+2} = 1 + \sum_{i=1}^{k} F_i$ for all $k \leq n$

$$\begin{aligned} k = n + 1: \qquad F_{n+3} &= 1 + \sum_{i=1}^{n+1} F_i \\ &= 1 + \sum_{i=1}^{n} F_i + F_{n+1} \\ &= F_{n+2} + F_{n+1} \end{aligned}$$

$\square$

Property 6.2 $F_{k+2} \geq \phi^k$

Proof:
[By induction]

$$\begin{aligned} k = 0: \qquad F_2 &= 1 \\ &\geq \phi^0 \\ k = 1: \qquad F_3 &= 2 \\ &\geq \phi = \frac{1 + \sqrt{5}}{2} = 1.618.. \end{aligned}$$

Assume $F_{k+2} \geq \phi^k$ for all $k < n$

$$
\begin{aligned}
k = n: \qquad F_{n+2} &= F_{n+1} + F_n \\
&\geq \phi^{n-1} + \phi^{n-2} \\
&\geq \frac{1+\phi}{\phi^2} \cdot \phi^n \\
&\geq \phi^n
\end{aligned}
$$

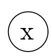
□

Theorem 6.1 *x is a root of any subtree. $size(x) \geq F_{k+2} \geq \phi^k$ where k is a degree of x*

Proof:

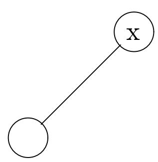
[By induction]

$k = 0$:



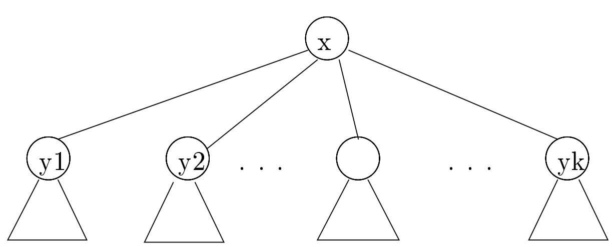
$size(x) = 1 \geq 1 \geq 1$

$k = 1$:



$size(x) = 2 \geq 2 \geq \frac{1+\sqrt{5}}{2}$

Assume $size(x) \geq F_{k+2} \geq \phi^k$ for any $k \leq n$



$k = n + 1$:

$$
\begin{aligned}
size(x) &= 1 + size(y_1) + ... + size(y_i) + ... + size(y_k) \\
&\geq 1 + 1 + 1 + F_3 + ... + F_k \;\; (from \;\; assumption) \\
&\geq 1 + F_1 + F_2 + ... + F_k \\
&\geq F_{k+2} \;\; (from \;\; property1) \\
&\geq \phi^k \;\; (from \;\; property2) \\
\log_\phi size(x) &\geq k
\end{aligned}
$$

So the number of children of node x is bounded by $\log_\phi size(x)$. □