CMSC 651 Advanced Algorithms	Lecture 8
Lecturer: Samir Khuller	Th. Feb 21, 2002

Original notes by Sibel Adali and Andrew Vakhutinsky.

15 Network Flow - Maximum Flow Problem

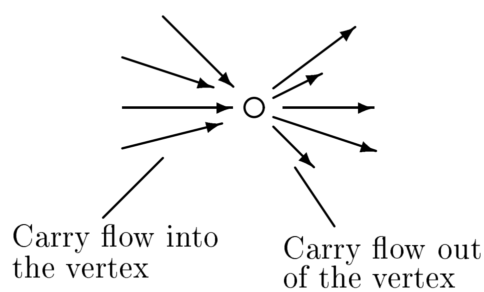
Read [21, 5, 19].

The problem is defined as follows: Given a directed graph $G^d = (V, E, s, t, c)$ where s and t are special vertices called the source and the sink, and c is a capacity function $c : E \rightarrow \Re^+$, find the maximum flow from s to t.

Flow is a function $f : E \rightarrow \Re$ that has the following properties:

1. **(Skew Symmetry)** $f(u, v) = -f(v, u)$

2. **(Flow Conservation)** $\Sigma_{v \in V} f(u, v) = 0$ for all $u \in V - \{s, t\}$.
   (Incoming flow) $\Sigma_{v \in V} f(v, u) =$ (Outgoing flow) $\Sigma_{v \in V} f(u, v)$



Flow Conservation

3. **(Capacity Constraint)** $f(u, v) \leq c(u, v)$

Maximum flow is the maximum value $|f|$ given by

$$|f| = \Sigma_{v \in V} f(s, v) = \Sigma_{v \in V} f(v, t).$$

Definition 15.1 (Residual Graph) *G_f^D is defined with respect to some flow function f, $G_f = (V, E_f, s, t, c')$ where $c'(u, v) = c(u, v) - f(u, v)$. Delete edges for which $c'(u, v) = 0$.*

As an example, if there is an edge in G from u to v with capacity 15 and flow 6, then in G_f there is an edge from u to v with capacity 9 (which means, I can still push 9 more units of liquid) and an edge from v to u with capacity 6 (which means, I can cancel all or part of the 6 units of liquid I'm currently pushing) [1]. E_f contains all the edges e such that $c'(e) > 0$.

Lemma 15.2 *Here are some easy to prove facts:*

1. *f' is a flow in G_f iff $f + f'$ is a flow in G.*

2. *f' is a maximum flow in G_f iff $f + f'$ is a maximum flow in G.*

3. *$|f + f'| = |f| + |f'|$.*

[1] Since there was no edge from v to u in G, then its capacity was 0 and the flow on it was -6. Then, the capacity of this edge in G_f is $0 - (-6) = 6$.

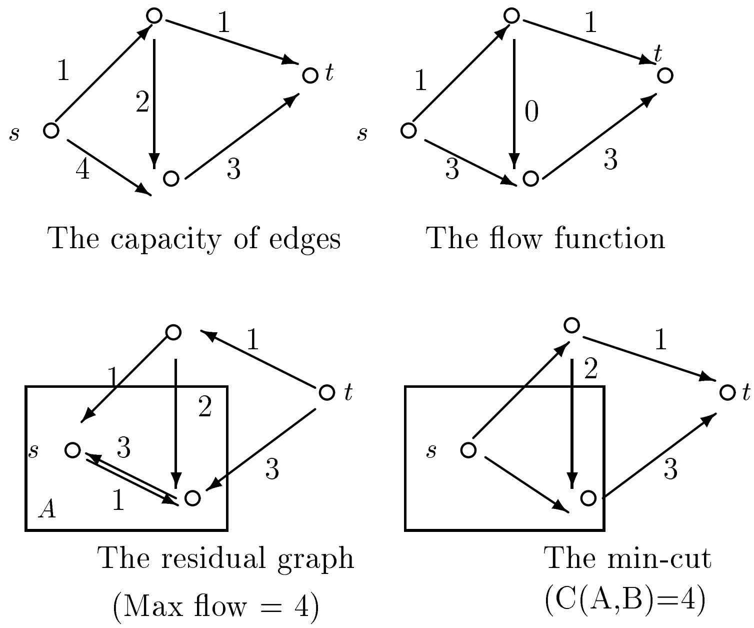


Figure 14: An example

4. If f is a flow in G, and f^* is the maximum flow in G, then $f^* - f$ is the maximum flow in G_f.

Definition 15.3 (Augmenting Path) *A path P from s to t in the residual graph G_f is called augmenting if for all edges (u, v) on P, $c'(u, v) > 0$. The* residual capacity *of an augmenting path P is $\min_{e \in P} c'(e)$.*

The idea behind this definition is that we can send a positive amount of flow along the augmenting path from s to t and "augment" the flow in G. (This flow increases the real flow on some edges and cancels flow on other edges, by reversing flow.)

Definition 15.4 (Cut) *An (s, t) cut is a partitioning of V into two sets A and B such that $A \cap B = \emptyset$, $A \cup B = V$ and $s \in A, t \in B$.*

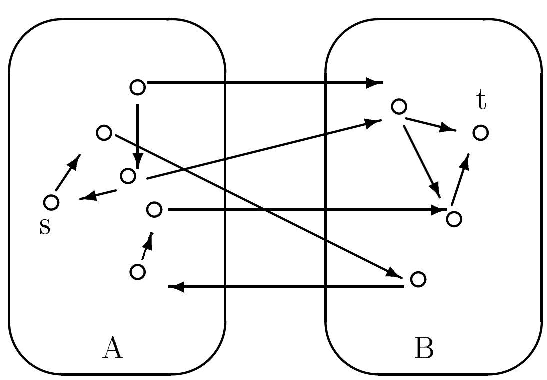


Figure 15: An (s, t) Cut

Definition 15.5 (Capacity Of A Cut) *The capacity $C(A, B)$ is given by*

$$C(A, B) = \Sigma_{a \in A, b \in B} \; c(a, b).$$

By the capacity constraint we have that $|f| = \Sigma_{v \in V} f(s,v) \leq C(A,B)$ for any (s,t) cut (A,B). Thus the capacity of the minimum capacity s,t cut is an upper bound on the value of the maximum flow.

Theorem 15.6 (Max flow - Min cut Theorem) *The following three statements are equivalent:*

1. *f is a maximum flow.*
2. *There exists an (s,t) cut (A,B) with $C(A,B) = |f|$.*
3. *There are no augmenting paths in G_f.*

An augmenting path is a directed path from s to t.

Proof:
We will prove that (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

((2) $\Rightarrow$ (1)**)** Since no flow can exceed the capacity of an (s,t) cut (i.e. $f(A,B) \leq C(A,B)$), the flow that satisfies the equality condition of **(2)** must be the maximum flow.

((1) $\Rightarrow$ (3)**)** If there was an augmenting path, then I could augment the flow and find a larger flow, hence f wouldn't be maximum.

((3) $\Rightarrow$ (2)**)** Consider the residual graph G_f. There is no directed path from s to t in G_f, since if there was this would be an augmenting path. Let $A = \{v | v$ is reachable from s in $G_f\}$. A and $\overline{A}$ form an (s,t) cut, where all the edges go from $\overline{A}$ to A. The flow f' must be equal to the capacity of the edge, since for all $u \in A$ and $v \in \overline{A}$, the capacity of (u,v) is 0 in G_f and $0 = c(u,v) - f'(u,v)$, therefore $c(u,v) = f'(u,v)$. Then, the capacity of the cut in the original graph is the total capacity of the edges from A to $\overline{A}$, and the flow is exactly equal to this amount. $\square$

A "Naive" Max Flow Algorithm:

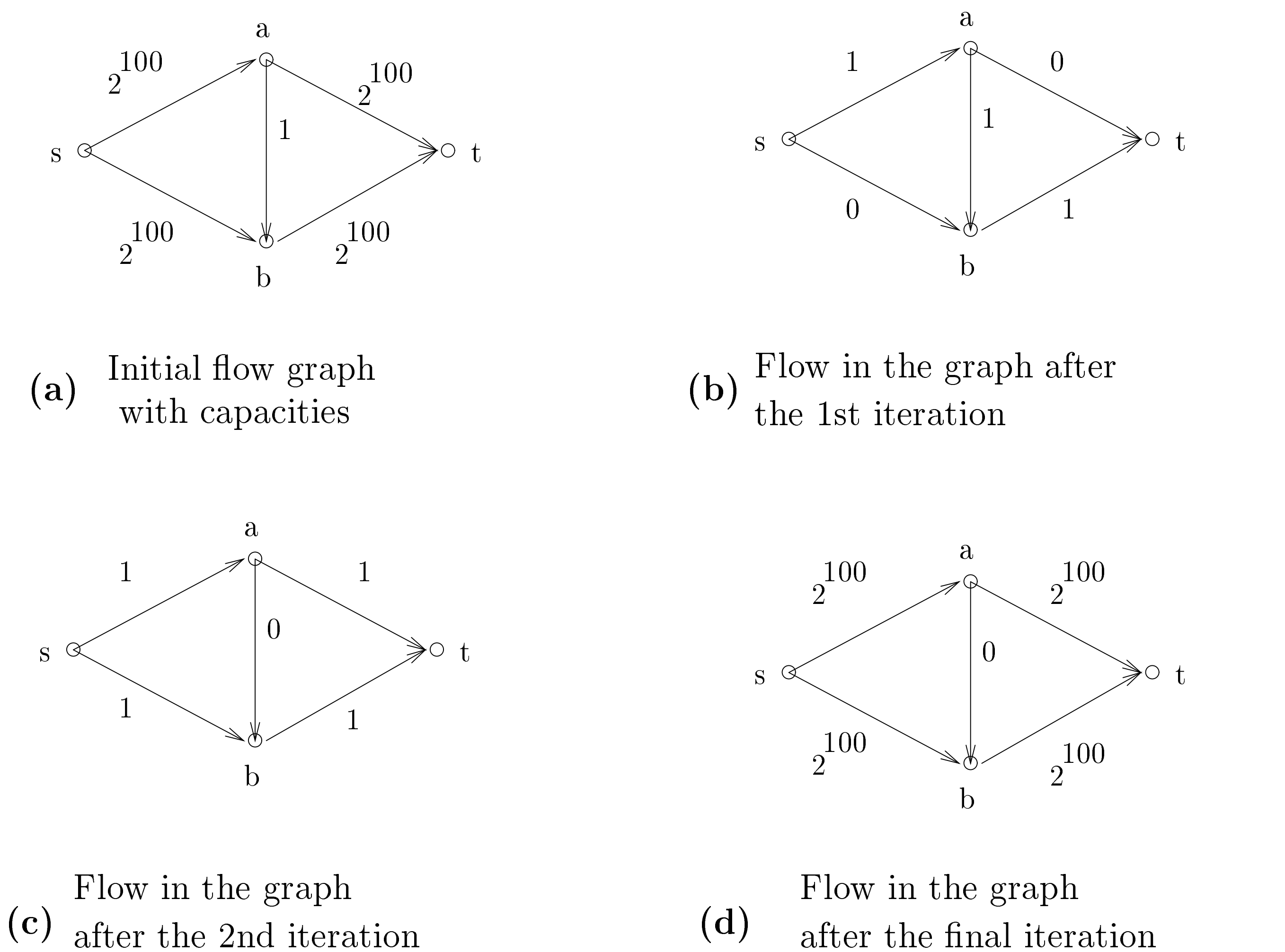
Initially let f be the 0 flow
while (there is an augmenting path P in G_f) **do**
    $c(P) \leftarrow \min_{e \in P} c'(e)$;
    send flow amount $c(P)$ along P;
    update flow value $|f| \leftarrow |f| + c(P)$;
    compute the new residual flow network G_f

Analysis: The algorithm starts with the zero flow, and stops when there are no augmenting paths from s to t. If all edge capacities are integral, the algorithm will send at least one unit of flow in each iteration (since we only retain those edges for which $c'(e) > 0$). Hence the running time will be $O(m|f^*|)$ in the worst case ($|f^*|$ is the value of the max-flow).

A worst case example. Consider a flow graph as shown on the Fig. 16. Using augmenting paths (s,a,b,t) and (s,b,a,t) alternatively at odd and even iterations respectively (fig.1(b-c)), it requires total $|f^*|$ iterations to construct the max flow.

If all capacities are rational, there are examples for which the flow algorithm might never terminate. The example itself is intricate, but this is a fact worth knowing.

Example. Consider the graph on Fig. 17 where all edges except (a,d), (b,e) and (c,f) are unbounded (have comparatively large capacities) and c$(a,d) = 1$, c$(b,e) = R$ and c$(c,f) = R^2$. Value R is chosen such that $R = \frac{\sqrt{5}-1}{2}$ and, clearly (for any $n \geq 0$, $R^{n+2} = R^n - R^{n+1}$. If augmenting paths are selected as shown on Fig. 17 by dotted lines, residual capacities of the edges (a,d), (b,e) and (c,f) will remain 0, R^{3k+1} and R^{3k+2} after every $(3k+1)$st iteration $(k = 0,1,2,\ldots)$. Thus, the algorithm will never terminate.



(a) Initial flow graph with capacities

(b) Flow in the graph after the 1st iteration

(c) Flow in the graph after the 2nd iteration

(d) Flow in the graph after the final iteration

Figure 16: Worst Case Example

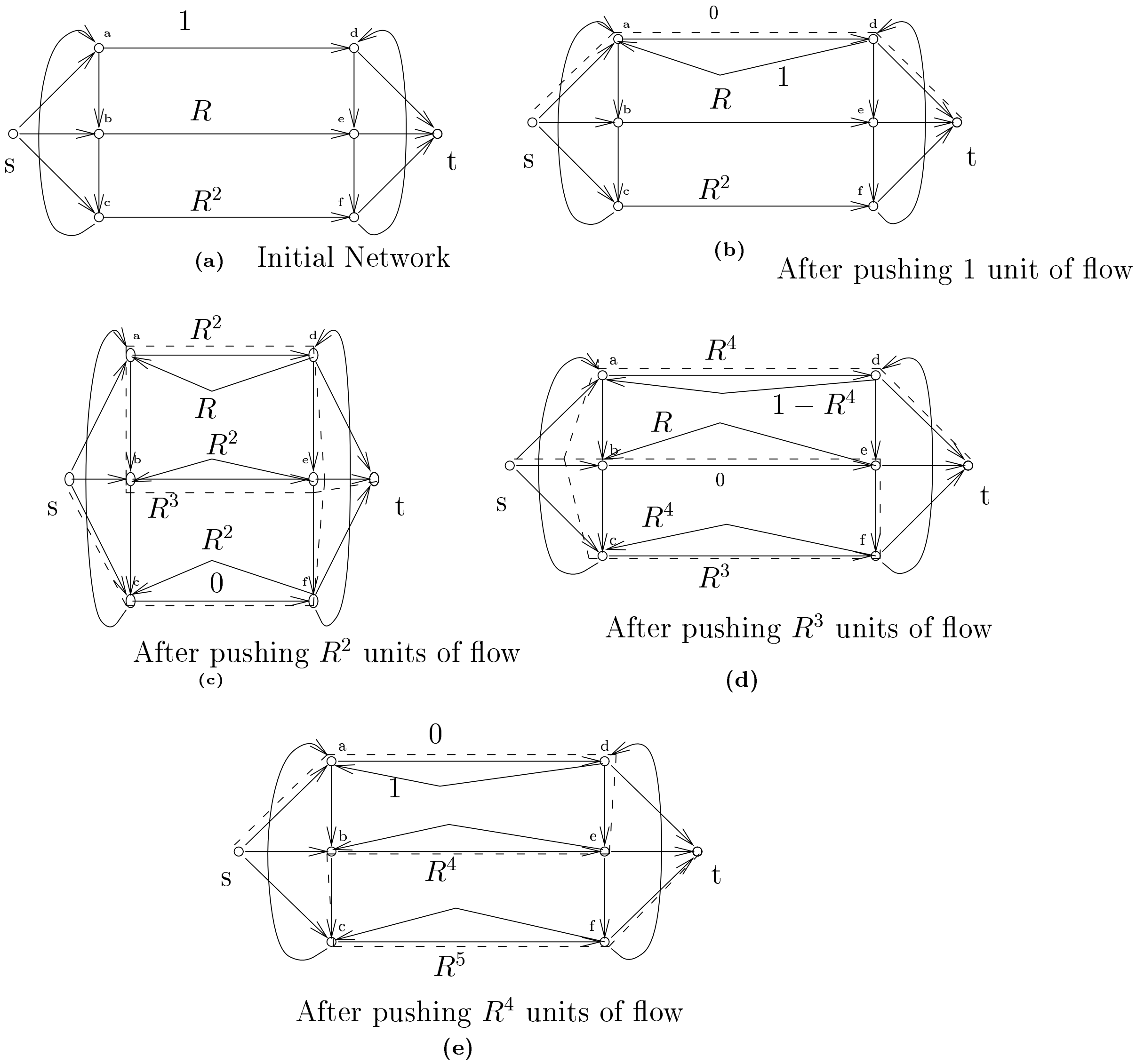


Figure 17: Non-terminating Example

CMSC 651 Advanced Algorithms	Lecture 09
Lecturer: Samir Khuller	Feb 26, 2002

Notes by Samir Khuller.

16 The Max Flow Problem

Today we will study the Edmonds-Karp algorithm that works when the capacities are integral, and has a much better running time than the Ford-Fulkerson method. (Edmonds and Karp gave a second heurisitc that we will study later.)

Assumption: Capacities are integers.

1st Edmonds-Karp Algorithm:

while (there is an augmenting path $s - t$ in G_f) **do**
  pick up an augmenting path (in G_f) with the highest residual capacity;
  use this path to augment the flow;

Analysis: We first prove that if there is a flow in G of value $|f|$, the highest capacity of an augmenting path in G_f is $\geq \frac{|f|}{m}$. In class we covered two different proofs of this lemma. The notion of flow decompositions is very useful so I am describing this proof in the notes.

Lemma 16.1 *Any flow in G can be expressed as a sum of at most m path flows in G and a flow in G of value 0, where m is the number of edges in G.*

Proof:
Let f be a flow in G. If $|f| = 0$, we are done. (We can assume that the flow on each edge is the same as the capacity of the edge, since the capacities can be artifically reduced without affecting the flow. As a result, edges that carry no flow have their capacities reduced to zero, and such edges can be discarded. The importance of this will become clear shortly.) Otherwise, let p be a path from s to t in the graph. Let $c(p) > 0$ be the bottleneck capacity of this path (edge with minimum flow/capacity). We can reduce the flow on this path by $c(p)$ and we output this flow path. The bottleneck edge now has zero capacity and can be deleted from the network, the capacities of all other edges on the path is lowered to reflect the new flow on the edge. We continue doing this until we are left with a zero flow ($|f| = 0$). Clearly, at most m paths are output during this procedure. □

Since the entire flow has been decomposed into at most m flow paths, there is at least one augmenting path with a capacity at least $\frac{|f|}{m}$.

Let the max flow value be f^*. In the first iteration we push at least $f_1 \geq \frac{f^*}{m}$ amount of flow. The value of the max-flow in the residual network (after one iteration) is at most

$$f^*(1 - 1/m).$$

The amount of flow pushed on the second iteration is at least

$$f_2 \geq (f^* - f_1)\frac{1}{m}.$$

The value of the max-flow in the residual network (after two iteations) is at most

$$f^* - f_1 - f_2 \leq f^* - f_1 - (f^* - f_1)\frac{1}{m} = f^*(1 - \frac{1}{m}) - f_1(1 - \frac{1}{m}) \leq f^*(1 - \frac{1}{m}) - \frac{f^*}{m}(1 - \frac{1}{m}).$$

$$= f^*(1 - \frac{1}{m})^2.$$

Finally, the max flow in the residual graph after the k^{th} iteration is

$$\leq f^*(\frac{m-1}{m})^k.$$

What is the smallest value of k that will reduce the max flow in the residual graph to 1?

$$f^*(\frac{m-1}{m})^k \leq 1 \ ,$$

Using the approximation $\log m - \log(m-1) = \Theta(\frac{1}{m})$ we can obtain a bound on k.

$$k = \Theta(m \log f^*).$$

This gives a bound on the number of iterations of the algorithm. Taking into a consideration that a path with the highest residual capacity can be picked up in time $O(m + n\log n)$, the overall time complexity of the algorithm is $O((m + n\log n)m \log f^*)$.

Tarjan's book gives a slightly different proof and obtains the same bound on the number of augmenting paths that are found by the algorithm.

History
Ford-Fulkerson (1956)
Edmonds-Karp (1969) $O(nm^2)$
Dinic (1970) $O(n^2m)$
Karzanov (1974) $O(n^3)$
Malhotra-Kumar-Maheshwari (1977) $O(n^3)$
Sleator-Tarjan (1980) $O(nm \log n)$
Goldberg-Tarjan (1986) $O(nm \log n^2/m)$