On the Science Behind Computing (Part I)

Samir Khuller *

1 More on Mathematical Induction

Let us consider two simple examples. Consider the following summation. (If you write out the sum of the first N cubes, then you can see that they are all perfect square, this is the first clue as to how we derived the guess for this sum.)

Say we want to prove something like the following: For all positive integers N

$$\sum_{i=1}^{N} i^3 = (\frac{N(N+1)}{2})^2$$

For example, for $N = 1$ the left side is just 1. The right side is $(\frac{1 \cdot 2}{2})^2 = 1$. So at least we know that this claim holds for $N = 1$. However this does not mean it will hold for any integer N. We can easily make up general forumlas that only hold for specific values (for example $2N - 1 = N$ for $N = 1$, but does not hold in general).

We can also check it for $N = 2$ quite easily since both sides are exactly 9. How do we confirm that it is true for any integer N? The secret lies in proving the following: suppose it holds for $N = k$, then we shall prove that it hold for $N = (k + 1)$. Now once we do that, if we can prove it hold for $N = 1$ then it follows that it holds for $N = 2$. But if it holds for $N = 2$, then it must hold for $N = 3$ etc. In this way we can see that it holds for all integers N.

Suppose it hold for $N = k$. Then $\sum_{i=1}^{k} i^3 = (\frac{k(k+1)}{2})^2$. We need to prove that $\sum_{i=1}^{k+1} i^3 = (\frac{(k+1)(k+2)}{2})^2$. But the left hand side is $\sum_{i=1}^{k} i^3 + (k + 1)^3$. Since we can assume that the sum of the first k cubes is $(\frac{k(k+1)}{2})^2$, we get $(\frac{k(k+1)}{2})^2 + (k + 1)^3 = \frac{k^2(k+1)^2+4(k+1)^3}{4}$. Simplifying gives $\frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4} = (\frac{(k+1)(k+2)}{2})^2$.

The second example we discuss is an unusual one (borrowed from the Discrete Math textbook by Liu). Suppose the USPS prints two kinds of stamps - 5c stamps and 3c stamps. We would like to show that for all $N \geq 8$, we can put exact postage using these two kinds of stamps. Note that while we can create 3 and 5 units of postage (using a single stamp) as well as 6 units of postage, there is no way to create 7 units of postage. Hence it is not at all obvious that we can get N units of postage for *all* values of $N \geq 8$.

We will do this proof by induction on N. Suppose we know how to make k units of postage. using x stamps of 3c each and y stamps of 5c each. If $y > 0$ then by simply removing one 5c stamp and replacing it by two 3c stamps, we can make $k + 1$ units of postage. If $y = 0$ then there are no 5c stamps in use to make k units. Since $k \geq 9$ we must be using at least three, 3c stamps. By removing three 3c stamps, and replacing them by two 5c stamps, we can get $k + 1$ units of postage.

This can now be turned into a scheme quite easily.

*Department of Computer Science, University of Maryland, College Park, MD 20742. E-mail : samir@cs.umd.edu.