Homework 2
Due in class on Wed, Feb 29, 2012

This short homework assignment will help you review your understanding of the simply typed lambda calculus, extended with integers:

\[ e ::= v \mid x \mid e \ e \]
\[ v ::= n \mid \lambda x. e \]
\[ t ::= \text{int} \mid t \to t \]
\[ A ::= \cdot \mid x : t, A \]

1. Here are small-step, call-by-value operational semantics for lambda calculus:

\[
\begin{align*}
\text{Beta} & \quad (\lambda x. e_1) v_2 \to e_1[x \mapsto v_2] \\
\text{Left} & \quad e_1 \to e'_1 \\
\text{Right} & \quad e_2 \to e'_2 \\
& \quad v v_2 \to v_2'
\end{align*}
\]

(a) Draw derivations showing that the following reductions hold:

i. \((\lambda x.42) 13 \to 42\)

ii. \(((\lambda x.x) (\lambda y.y)) (\lambda z.z) \to (\lambda y.y) (\lambda z.z)\)

iii. \((\lambda x.x) ((\lambda x.\lambda y.x) 42) \to (\lambda x.x) (\lambda y.42)\)

(b) Give a reduction that is possible in the fully non-deterministic operational semantics for lambda calculus, but which is not possible in the call-by-value operational semantics.

2. Here are the type rules for the simply typed lambda calculus:

\[
\begin{align*}
\text{INT} & \quad x \in \text{dom}(A) \\
\text{VAR} & \quad x : t, A \vdash e : t' \\
\text{LAM} & \quad A \vdash \lambda x. e : t \to t' \\
\text{APP} & \quad A \vdash e_1 : t' \to t' \quad A \vdash e_2 : t
\end{align*}
\]

(a) Draw derivations showing that the following typing judgments hold:

i. \(\cdot \vdash 42 : \text{int}\)

ii. \(y : \text{int} \vdash \lambda x.y : \text{int} \to \text{int}\)

iii. \(\cdot \vdash \lambda x.\lambda y.x : \text{int} \to \text{int} \to \text{int}\)

iv. \(+ : \text{int} \to \text{int} \to \text{int} \vdash (\lambda f.42) (\lambda x.+x.3) : \text{int}\)

(b) Write down an expression in the simply typed lambda calculus with integers that does not get “stuck” in under the operational semantics in part 1 (meaning, it will either run forever or reduce to a value), and yet is ill-typed. (Note that you do not have if...then...else available to you in this language as a built-in.)

3. By giving a counterexample, show that the following statement, which is essentially the reverse of Preservation, is false: If \(A \vdash e' : t\) and \(e \to e'\), then \(A \vdash e : t\).