# CMSC 330: Organization of Programming Languages

## Lambda Calculus

# Programming Language Features

- Many features exist simply for convenience
  - Multi-argument functions  foo ( a, b, c )
    - Use currying or tuples
  - Loops  while (a < b) …
    - Use recursion
  - Side effects  a := 1
    - Use functional programming

- So what language features are really needed?

# Turing Completeness

- Computational system that can
  - Simulate a Turing machine
  - Compute every Turing-computable function
- A programming language is Turing complete if
  - It can map every Turing machine to a program
  - A program can be written to emulate a Turing machine
  - It is a superset of a known Turing-complete language
- Most powerful programming language possible
  - Since Turing machine is most powerful automaton

# Programming Language Theory

- Come up with a "core" language
  - That's as small as possible
  - But still Turing complete
- Helps illustrate important
  - Language features
  - Algorithms
- One solution
  - Lambda calculus

# Lambda Calculus (λ-calculus)

- Proposed in 1930s by
  - Alonzo Church
    (born in Washingon DC!)
- Formal system
  - Designed to investigate functions & recursion
  - For exploration of foundations of mathematics
- Now used as
  - Tool for investigating computability
  - Basis of functional programming languages
    - Lisp, Scheme, ML, OCaml, Haskell…

# Lambda Expressions

- A lambda calculus expression is defined as

$$
\begin{array}{lll}
e ::= & x & \text{variable} \\
 & \mid\ \lambda x.e & \text{function} \\
 & \mid\ e\ e & \text{function application}
\end{array}
$$

- $\lambda x.e$ is like `(fun x -> e)` in OCaml
- That's it! Nothing but higher-order functions

# Three Conveniences

- Syntactic sugar for local declarations
  - let x = e1 in e2 is short for (λx.e2) e1
- Scope of λ extends as far right as possible
  - Subject to scope delimited by parentheses
  - λx. λy.x y is same as λx.(λy.(x y))
- Function application is left-associative
  - x y z is (x y) z
  - Same rule as OCaml

# Lambda Calculus Semantics

- All we've got are functions
  - So all we can do is call them
- To evaluate (λx.e1) e2
  - Evaluate e1 with x replaced by e2
- This application is called beta-reduction
  - (λx.e1) e2 → e1[x:=e2]
    - e1[x:=e2] is e1 with occurrences of x replaced by e2
    - This operation is called *substitution*
    - Slightly different than the environments we saw for Ocaml
      - Do syntactic substitutions to replace formals with actuals
      - Instead of using environment to map formals to actuals
  - We allow reductions to occur anywhere in a term

# Beta Reduction Example

- $(\lambda x.\lambda z.x\ z)\ y$
  - $\rightarrow (\lambda x.(\lambda z.(x\ z)))\ y$ // since λ extends to right
  - $\rightarrow (\lambda x.(\lambda z.(x\ z)))\ y$ // apply $(\lambda x.e1)\ e2 \rightarrow e1[x:=e2]$
    // where $e1 = \lambda z.(x\ z)$, $e2 = y$
  - $\rightarrow \lambda z.(y\ z)$ // final result

Parameters
- Formal
- Actual

- Equivalent OCaml code
  - (fun x -> (fun z -> (x z))) y → fun z -> (y z)

# Lambda Calculus Examples

- $(\lambda x.x)\ z \rightarrow z$
- $(\lambda x.y)\ z \rightarrow y$
- $(\lambda x.x\ y)\ z \rightarrow z\ y$
  - A function that applies its argument to y

# Lambda Calculus Examples (cont.)

- (λx.x y) (λz.z) → (λz.z) y → y
- (λx.λy.x y) z → λy.z y
  - A curried function of two arguments
  - Applies its first argument to its second
- (λx.λy.x y) (λz.zz) x →(λy.(λz.zz)y)x → (λz.zz)x → xx

# Defining Substitution

- Use recursion on structure of terms!
  - x[x:=e] = e        // Replace x by e
  - y[x:=e] = y        // y is different than x, so no effect
  - (e1 e2)[x:=e] = (e1[x:=e]) (e2[x:=e])
    // Substitute both parts of application
  - (λx.e’)[x:=e] = λx.e’
    - In λx.e’, the x is a parameter, and thus a local variable that is different from other x’s.
    - So the substitution has no effect in this case, since the x being substituted for is different from the parameter x that is in e’!
  - (λy.e’)[x:=e] = ?
    - The parameter y does not share the same name as x, the variable being substituted for
    - Is λy.(e’ [x:=e]) correct?

# Static Scoping & Alpha Conversion

- Lambda calculus uses static scoping
- Consider the following
  - $(\lambda x.x\ (\lambda x.x))\ z \rightarrow ?$
    - The rightmost "x" refers to the second binding
  - This is a function that
    - Takes its argument and applies it to the identity function
- This function is "the same" as $(\lambda x.x\ (\lambda y.y))$
  - Renaming bound variables consistently is allowed
    - This is called alpha-renaming or alpha conversion
  - Ex. $\lambda x.x = \lambda y.y = \lambda z.z \qquad \lambda y.\lambda x.y = \lambda z.\lambda x.z$

# Static Scoping (cont.)

- How about the following?
  - (λx.λy.x y) y → ?
  - When we replace y inside, we don't want it to be captured by the inner binding of y, as this violates static scoping
  - I.e., (λx.λy.x y) y ≠ λy.y y
- Solution
  - (λx.λy.x y) is "the same" as (λx.λz.x z)
    - Due to alpha conversion
  - So change (λx.λy.x y) y to (λx.λz.x z) y first
    - Now (λx.λz.x z) y → λz.y z

# Completing the Definition of Substitution

- Recall: we need to define $(\lambda y.e')[x:=e]$
  - We want to avoid capturing (free) occurrences of y in e
  - Solution: alpha-conversion!
    - Change y to a variable w that does not appear in e' or e. (Such a w is called fresh.)
    - Replace all occurrences of y in e' by w.
    - Then replace all occurrences of x in e' by e!
- Formally:

  $(\lambda y.e')[x:=e] = \lambda w.((e'\ [y:=w])\ [x:=e])$ (w is fresh)

# Beta-Reduction, Again

- Whenever we do a step of beta reduction
  - $(\lambda x.e1)\ e2 \to e1[x{:=}e2]$
  - We must alpha-convert variables as necessary
  - Usually performed implicitly (w/o showing conversion)

- Examples
  - $(\lambda x.\lambda y.x\ y)\ y = (\lambda x.\lambda z.x\ z)\ y \to \lambda z.y\ z$ // $y \to z$
  - $(\lambda x.x\ (\lambda x.x))\ z = (\lambda y.y\ (\lambda x.x))\ z \to z\ (\lambda x.x)$ // $x \to y$
  - $(\lambda x.x\ (\lambda x.x))\ z = (\lambda x.x\ (\lambda y.y))\ z \to z\ (\lambda y.y)$ // $x \to y$

# Encodings

- The lambda calculus is Turing complete

- Means we can encode any computation we want
  - If we're sufficiently clever...

- Examples
  - Booleans
  - Pairs
  - Natural numbers & arithmetic
  - Looping

# Booleans

- Church's encoding of mathematical logic
  - true = λx.λy.x
  - false = λx.λy.y
  - if a then b else c
    - Defined to be the λ expression: a b c

- Examples
  - if true then b else c → (λx.λy.x) b c → (λy.b) c → b
  - if false then b else c → (λx.λy.y) b c → (λy.y) c → c

# Booleans (cont.)

- Other Boolean operations
  - not = λx.((x false) true)
    - not x = if x then false else true!
    - not true → (λx.(x false) true) true → ((true false) true) → false
  - and = λx.λy.((x y) false)
    - and x y = if x then y else false
  - or = λx.λy.((x true) y)
    - or x y = if x then true else y
- Given these operations
  - Can build up a logical inference system

# Pairs

- Encoding of a pair a, b
  - (a,b) = λx.if x then a else b
  - fst = λf.f true
  - snd = λf.f false
- Examples
  - fst (a,b) = (λf.f true) (λx.if x then a else b) →
    (λx.if x then a else b) true →
    if true then a else b → a
  - snd (a,b) = (λf.f false) (λx.if x then a else b) →
    (λx.if x then a else b) false →
    if false then a else b → b

# Natural Numbers (Church* Numerals)

- Encoding of non-negative integers
  - 0 = λf.λy.y
  - 1 = λf.λy.f y
  - 2 = λf.λy.f (f y)
  - 3 = λf.λy.f (f (f y))

    i.e., n = λf.λy.<apply f n times to y>
  - Formally: n+1 = λf.λy.f (n f y)

*(Alonzo Church, of course)

# Operations On Church Numerals

- Successor
  - succ = λz.λf.λy.f (z f y)
  - 0 = λf.λy.y
  - 1 = λf.λy.f y
- Example
  - succ 0 =

    (λz.λf.λy.f (z f y)) (λf.λy.y) →

    λf.λy.f ((λf.λy.y) f y) →

    λf.λy.f ((λy.y) y) →

    λf.λy.f y

    = 1

Since (λx.y) z → y

# Operations On Church Numerals (cont.)

- IsZero?
  - iszero = λz.z (λy.false) true

    This is equivalent to λz.((z (λy.false)) true)

- Example
  - iszero 0 =

    (λz.z (λy.false) true) (λf.λy.y) →

    (λf.λy.y) (λy.false) true →

    (λy.y) true →          Since (λx.y) z → y

    true
  - 0 = λf.λy.y

# Arithmetic Using Church Numerals

- If M and N are numbers (as λ expressions)
  - Can also encode various arithmetic operations
- Addition
  - $M + N = \lambda x.\lambda y.(M\ x)((N\ x)\ y)$

    Equivalently: $+ = \lambda M.\lambda N.\lambda x.\lambda y.(M\ x)((N\ x)\ y)$
    - In prefix notation (+ M N)
- Multiplication
  - $M * N = \lambda x.(M\ (N\ x))$

    Equivalently: $* = \lambda M.\lambda N.\lambda x.(M\ (N\ x))$
    - In prefix notation (* M N)

# Arithmetic (cont.)

- $1 = \lambda f.\lambda y.f\ y$
- $2 = \lambda f.\lambda y.f\ (f\ y)$

- Prove 1+1 = 2
  - $1+1 = \lambda x.\lambda y.(1\ x)((1\ x)\ y) =$
  - $\lambda x.\lambda y.((\lambda x.\lambda y.x\ y)\ x)(((\lambda x.\lambda y.x\ y)\ x)\ y) \rightarrow$
  - $\lambda x.\lambda y.(\lambda y.x\ y)(((\lambda x.\lambda y.x\ y)\ x)\ y) \rightarrow$
  - $\lambda x.\lambda y.(\lambda y.x\ y)((\lambda y.x\ y)\ y) \rightarrow$
  - $\lambda x.\lambda y.x\ ((\lambda y.x\ y)\ y) \rightarrow$
  - $\lambda x.\lambda y.x\ (x\ y) = 2$

Many implicit alpha conversions

- With these definitions
  - Can build a theory of arithmetic

# Looping

- Define D = λx.x x, then
  - D D = (λx.x x) (λx.x x) → (λx.x x) (λx.x x) = D D

- So D D is an infinite loop
  - In general, self application is how we get looping

# The Fixpoint Combinator

Y = λf.(λx.f (x x)) (λx.f (x x))

- Then

  Y F =

  (λf.(λx.f (x x)) (λx.f (x x))) F →

  (λx.F (x x)) (λx.F (x x)) →

  F ((λx.F (x x)) (λx.F (x x)))

  = F (Y F)


- Y F is a *fixed point* (aka "fixpoint") of F
- Thus Y F = F (Y F) = F (F (Y F)) = ...
  - We can use Y to achieve recursion for F

# Example

fact = $\lambda$f. $\lambda$n.if n = 0 then 1 else n * (f (n-1))

- The second argument to fact is the integer
- The first argument is the function to call in the body
  - We'll use Y to make this recursively call fact

(Y fact) 1 = (fact (Y fact)) 1

→ if 1 = 0 then 1 else 1 * ((Y fact) 0)

→ 1 * ((Y fact) 0)

→ 1 * (fact (Y fact) 0)

→ 1 * (if 0 = 0 then 1 else 0 * ((Y fact) (-1))

→ 1 * 1 → 1

# Discussion

- Lambda calculus is Turing-complete
  - Most powerful language possible
  - Can represent pretty much anything in "real" language
    - Using clever encodings
- But programs would be
  - Pretty slow (10000 + 1 → thousands of function calls)
  - Pretty large (10000 + 1 → hundreds of lines of code)
  - Pretty hard to understand (recognize 10000 vs. 9999)
- In practice
  - We use richer, more expressive languages
  - That include built-in primitives

# The Need For Types

- Consider the untyped lambda calculus
  - false = $\lambda x.\lambda y.y$
  - 0 = $\lambda x.\lambda y.y$
- Since everything is encoded as a function...
  - We can easily misuse terms…
    - false 0 $\rightarrow \lambda y.y$
    - if 0 then ...

  …because everything evaluates to some function
- The same thing happens in assembly language
  - Everything is a machine word (a bunch of bits)
  - All operations take machine words to machine words

# Simply-Typed Lambda Calculus

- $e ::= n \mid x \mid \lambda x{:}t.e \mid e\ e$
  - Added integers n as primitives
    - Need at least two distinct types (integer & function)…
    - …to have type errors
  - Functions now include the type of their argument

# Simply-Typed Lambda Calculus (cont.)

- t ::= int | t → t
  - int is the type of integers
  - t1 → t2 is the type of a function
    - That takes arguments of type t1 and returns result of type t2
  - t1 is the domain and t2 is the range
  - Notice this is a recursive definition
    - So we can give types to higher-order functions

# Summary

- Lambda calculus shows issues with
  - Scoping
  - Higher-order functions
  - Types
- Useful for understanding how languages work