

CMSC 330: Organization of Programming Languages

Lambda Calculus

Programming Language Features

- ▶ Many features exist simply for convenience

- Multi-argument functions `foo ( a, b, c )`

- Use currying or tuples

- Loops `while (a < b) ...`

- Use recursion

- Side effects `a := 1`

- Use functional programming

- ▶ So what language features are really needed?

Turing Completeness

- ▶ Computational system that can
 - Simulate a Turing machine
 - Compute every Turing-computable function
- ▶ A programming language is **Turing complete** if
 - It can map every Turing machine to a program
 - A program can be written to emulate a Turing machine
 - It is a superset of a known Turing-complete language
- ▶ Most powerful programming language possible
 - Since Turing machine is most powerful automaton

Programming Language Theory

- ▶ Come up with a “core” language
 - That's as small as possible
 - But still Turing complete
- ▶ Helps illustrate important
 - Language features
 - Algorithms
- ▶ One solution
 - Lambda calculus

Lambda Calculus (λ -calculus)

- ▶ Proposed in 1930s by
 - Alonzo Church
(born in Washington DC!)
- ▶ Formal system
 - Designed to investigate functions & recursion
 - For exploration of foundations of mathematics
- ▶ Now used as
 - Tool for investigating computability
 - Basis of functional programming languages
 - Lisp, Scheme, ML, OCaml, Haskell...



Lambda Expressions

- ▶ A lambda calculus **expression** is defined as

$e ::= x$

variable

| $\lambda x.e$

function

| $e e$

function application

- ▶ $\lambda x.e$ is like `(fun x -> e)` in OCaml
- ▶ That's it! Nothing but higher-order functions

Three Conveniences

- ▶ Syntactic sugar for local declarations
 - `let x = e1 in e2` is short for `( $\lambda x.e2$ ) e1`
- ▶ Scope of λ extends as far right as possible
 - Subject to scope delimited by parentheses
 - `$\lambda x. \lambda y.x y$` is same as `$\lambda x. (\lambda y.(x y))$`
- ▶ Function application is left-associative
 - `x y z` is `(x y) z`
 - Same rule as OCaml

Lambda Calculus Semantics

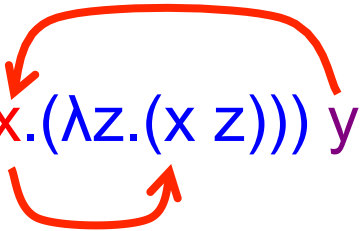
- ▶ All we've got are functions
 - So all we can do is call them
- ▶ To evaluate $(\lambda x.e1) e2$
 - Evaluate $e1$ with x replaced by $e2$
- ▶ This application is called **beta-reduction**
 - $(\lambda x.e1) e2 \rightarrow e1[x:=e2]$
 - $e1[x:=e2]$ is $e1$ with occurrences of x replaced by $e2$
 - This operation is called *substitution*
 - Slightly different than the environments we saw for Ocaml
 - Do syntactic substitutions to replace formals with actuals
 - Instead of using environment to map formals to actuals
 - We allow reductions to occur anywhere in a term

Beta Reduction Example

► $(\lambda x. \lambda z. x z) y$

$\rightarrow (\lambda x. (\lambda z. (x z))) y$

// since λ extends to right

 $\rightarrow (\lambda x. (\lambda z. (x z))) y$

// apply $(\lambda x. e1) e2 \rightarrow e1[x:=e2]$

// where $e1 = \lambda z. (x z)$, $e2 = y$

$\rightarrow \lambda z. (y z)$

// final result

Parameters

• Formal

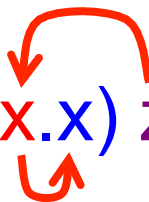
• Actual

► Equivalent OCaml code

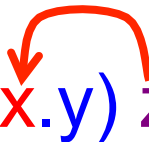
• $(\text{fun } x \rightarrow (\text{fun } z \rightarrow (x z))) y \rightarrow \text{fun } z \rightarrow (y z)$

Lambda Calculus Examples

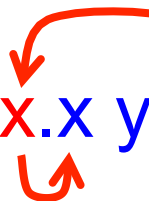
► $(\lambda x.x) z \rightarrow z$



► $(\lambda x.y) z \rightarrow y$



► $(\lambda x.x y) z \rightarrow z y$



- A function that applies its argument to y

Lambda Calculus Examples (cont.)

► $(\lambda x.x \ y) (\lambda z.z) \rightarrow (\lambda z.z) \ y \rightarrow y$

► $(\lambda x.\lambda y.x \ y) \ z \rightarrow \lambda y.z \ y$

- A curried function of two arguments
- Applies its first argument to its second

► $(\lambda x.\lambda y.x \ y) (\lambda z.zz) \ x \rightarrow (\lambda y.(\lambda z.zz) \ y) \ x \rightarrow (\lambda z.zz) \ x \rightarrow xx$

Defining Substitution

- ▶ Use recursion on structure of terms!
 - $x[x:=e] = e$ // Replace x by e
 - $y[x:=e] = y$ // y is different than x , so no effect
 - $(e1\ e2)[x:=e] = (e1[x:=e])\ (e2[x:=e])$
// Substitute both parts of application
 - $(\lambda x.e')[x:=e] = \lambda x.e'$
 - In $\lambda x.e'$, the x is a parameter, and thus a local variable that is different from other x 's.
 - So the substitution has no effect in this case, since the x being substituted for is different from the parameter x that is in e' !
 - $(\lambda y.e')[x:=e] = ?$
 - The parameter y does not share the same name as x , the variable being substituted for
 - Is $\lambda y.(e'[x:=e])$ correct?

Static Scoping & Alpha Conversion

- ▶ Lambda calculus uses **static scoping**
- ▶ Consider the following
 - $(\lambda x.x (\lambda x.x)) z \rightarrow ?$
 - The rightmost “x” refers to the second binding
 - This is a function that
 - Takes its argument and applies it to the identity function
- ▶ This function is “the same” as $(\lambda x.x (\lambda y.y))$
 - Renaming bound variables consistently is allowed
 - This is called **alpha-renaming** or **alpha conversion**
 - Ex. $\lambda x.x = \lambda y.y = \lambda z.z$ $\lambda y.\lambda x.y = \lambda z.\lambda x.z$

Static Scoping (cont.)

► How about the following?

- $(\lambda x. \lambda y. x \ y) \ y \rightarrow ?$
- When we replace y inside, we don't want it to be **captured** by the inner binding of y , as this violates static scoping
- I.e., $(\lambda x. \lambda y. x \ y) \ y \neq \lambda y. y \ y$

► Solution

- $(\lambda x. \lambda y. x \ y)$ is “the same” as $(\lambda x. \lambda z. x \ z)$
 - Due to alpha conversion
- So change $(\lambda x. \lambda y. x \ y) \ y$ to $(\lambda x. \lambda z. x \ z) \ y$ first
 - Now $(\lambda x. \lambda z. x \ z) \ y \rightarrow \lambda z. y \ z$

Completing the Definition of Substitution

- ▶ Recall: we need to define $(\lambda y.e')[x:=e]$
 - We want to avoid capturing (free) occurrences of y in e
 - Solution: alpha-conversion!
 - Change y to a variable w that does not appear in e' or e . (Such a w is called **fresh**.)
 - Replace all occurrences of y in e' by w .
 - Then replace all occurrences of x in e' by e !
- ▶ Formally:
$$(\lambda y.e')[x:=e] = \lambda w.((e' [y:=w]) [x:=e]) \text{ (} w \text{ is fresh)}$$

Beta-Reduction, Again

- ▶ Whenever we do a step of beta reduction
 - $(\lambda x.e1) e2 \rightarrow e1[x:=e2]$
 - We must alpha-convert variables as necessary
 - Usually performed implicitly (w/o showing conversion)
- ▶ Examples
 - $(\lambda x.\lambda y.x\ y) y = (\lambda x.\lambda z.x\ z) y \rightarrow \lambda z.y\ z \quad //\ y \rightarrow z$
 - $(\lambda x.x\ (\lambda x.x)) z = (\lambda y.y\ (\lambda x.x)) z \rightarrow z\ (\lambda x.x) \quad //\ x \rightarrow y$
 - $(\lambda x.x\ (\lambda x.x)) z = (\lambda x.x\ (\lambda y.y)) z \rightarrow z\ (\lambda y.y) \quad //\ x \rightarrow y$

Encodings

- ▶ The lambda calculus is Turing complete
- ▶ Means we can **encode** any computation we want
 - If we're sufficiently clever...
- ▶ Examples
 - Booleans
 - Pairs
 - Natural numbers & arithmetic
 - Looping

Booleans

- ▶ Church's encoding of mathematical logic

- $\text{true} = \lambda x. \lambda y. x$
- $\text{false} = \lambda x. \lambda y. y$
- if a then b else c
 - Defined to be the λ expression: $a \ b \ c$

- ▶ Examples

- if true then b else c $\rightarrow (\lambda x. \lambda y. x) \ b \ c \rightarrow (\lambda y. b) \ c \rightarrow b$ 
- if false then b else c $\rightarrow (\lambda x. \lambda y. y) \ b \ c \rightarrow (\lambda y. y) \ c \rightarrow c$ 

Booleans (cont.)

► Other Boolean operations

- $\text{not} = \lambda x.((x \text{ false}) \text{ true})$
 - $\text{not } x = \text{if } x \text{ then false else true!}$
 - $\text{not true} \rightarrow (\lambda x.(x \text{ false}) \text{ true}) \text{ true} \rightarrow ((\text{true false}) \text{ true}) \rightarrow \text{false}$
- $\text{and} = \lambda x.\lambda y.((x \ y) \text{ false})$
 - $\text{and } x \ y = \text{if } x \text{ then } y \text{ else false}$
- $\text{or} = \lambda x.\lambda y.((x \text{ true}) \ y)$
 - $\text{or } x \ y = \text{if } x \text{ then true else } y$

► Given these operations

- Can build up a logical inference system

Pairs

- ▶ Encoding of a pair a, b

- $(a,b) = \lambda x. \text{if } x \text{ then } a \text{ else } b$
- $\text{fst} = \lambda f. f \text{ true}$
- $\text{snd} = \lambda f. f \text{ false}$

- ▶ Examples

- $\text{fst } (a,b) = (\lambda f. f \text{ true}) (\lambda x. \text{if } x \text{ then } a \text{ else } b) \rightarrow$
 $(\lambda x. \text{if } x \text{ then } a \text{ else } b) \text{ true} \rightarrow$
 $\text{if true then } a \text{ else } b \rightarrow a$
- $\text{snd } (a,b) = (\lambda f. f \text{ false}) (\lambda x. \text{if } x \text{ then } a \text{ else } b) \rightarrow$
 $(\lambda x. \text{if } x \text{ then } a \text{ else } b) \text{ false} \rightarrow$
 $\text{if false then } a \text{ else } b \rightarrow b$

Natural Numbers (Church* Numerals)

► Encoding of non-negative integers

- $0 = \lambda f. \lambda y. y$
- $1 = \lambda f. \lambda y. f\ y$
- $2 = \lambda f. \lambda y. f\ (f\ y)$
- $3 = \lambda f. \lambda y. f\ (f\ (f\ y))$
- i.e., $n = \lambda f. \lambda y. \text{<apply } f \text{ } n \text{ times to } y\text{>}$
- Formally: $n+1 = \lambda f. \lambda y. f\ (n\ f\ y)$

*(Alonzo Church, of course)

Operations On Church Numerals

► Successor

- $\text{succ} = \lambda z. \lambda f. \lambda y. f (z f y)$

- $0 = \lambda f. \lambda y. y$

- $1 = \lambda f. \lambda y. f y$

► Example

- $\text{succ } 0 =$

$$(\lambda z. \lambda f. \lambda y. f (z f y)) (\lambda f. \lambda y. y) \rightarrow$$

$$\lambda f. \lambda y. f ((\lambda f. \lambda y. y) f y) \rightarrow$$

$$\lambda f. \lambda y. f ((\lambda y. y) y) \rightarrow$$

$$\lambda f. \lambda y. f y$$

$$= 1$$

Since $(\lambda x. y) z \rightarrow y$

Operations On Church Numerals (cont.)

► IsZero?

- $\text{iszero} = \lambda z.z (\lambda y.\text{false}) \text{true}$

This is equivalent to $\lambda z.((z (\lambda y.\text{false})) \text{true})$

► Example

- $\text{iszero } 0 =$

$(\lambda z.z (\lambda y.\text{false}) \text{true}) (\lambda f.\lambda y.y) \rightarrow$

$(\lambda f.\lambda y.y) (\lambda y.\text{false}) \text{true} \rightarrow$

$(\lambda y.y) \text{true} \rightarrow$

true

Since $(\lambda x.y) z \rightarrow y$

- $0 = \lambda f.\lambda y.y$

Arithmetic Using Church Numerals

- ▶ If M and N are numbers (as λ expressions)
 - Can also encode various arithmetic operations
- ▶ Addition
 - $M + N = \lambda x. \lambda y. (M\ x)((N\ x)\ y)$
Equivalently: $+ = \lambda M. \lambda N. \lambda x. \lambda y. (M\ x)((N\ x)\ y)$
 - In prefix notation (+ M N)
- ▶ Multiplication
 - $M * N = \lambda x. (M\ (N\ x))$
Equivalently: $* = \lambda M. \lambda N. \lambda x. (M\ (N\ x))$
 - In prefix notation (* M N)

Arithmetic (cont.)

► Prove $1+1 = 2$

- $1+1 = \lambda x.\lambda y.(1\ x)((1\ x)\ y) =$
- $\lambda x.\lambda y.((\lambda x.\lambda y.x\ y)\ x)((\lambda x.\lambda y.x\ y)\ x)\ y) \rightarrow$
- $\lambda x.\lambda y.(\lambda y.x\ y)((\lambda x.\lambda y.x\ y)\ x)\ y) \rightarrow$
- $\lambda x.\lambda y.(\lambda y.x\ y)((\lambda y.x\ y)\ y) \rightarrow$
- $\lambda x.\lambda y.x\ ((\lambda y.x\ y)\ y) \rightarrow$
- $\lambda x.\lambda y.x\ (x\ y) = 2$

Many implicit alpha conversions

► With these definitions

- Can build a theory of arithmetic

Looping

- ▶ Define $D = \lambda x. x x$, then
 - $D D = (\lambda x. x x) (\lambda x. x x) \rightarrow (\lambda x. x x) (\lambda x. x x) = D D$
- ▶ So $D D$ is an infinite loop
 - In general, self application is how we get looping

The Fixpoint Combinator

$$Y = \lambda f.(\lambda x.f (x x)) (\lambda x.f (x x))$$

► Then

$$Y F =$$

$$(\lambda f.(\lambda x.f (x x)) (\lambda x.f (x x))) F \rightarrow$$

$$(\lambda x.F (x x)) (\lambda x.F (x x)) \rightarrow$$

$$F ((\lambda x.F (x x)) (\lambda x.F (x x)))$$

$$= F (Y F)$$



► $Y F$ is a *fixed point* (aka “fixpoint”) of F

► Thus $Y F = F (Y F) = F (F (Y F)) = \dots$

- We can use Y to achieve recursion for F

Example

$\text{fact} = \lambda f. \lambda n. \text{if } n = 0 \text{ then } 1 \text{ else } n * (f (n-1))$

- The second argument to fact is the integer
- The first argument is the function to call in the body
 - We'll use Y to make this recursively call fact

$(Y \text{ fact}) 1 = (\text{fact } (Y \text{ fact})) 1$

$\rightarrow \text{if } 1 = 0 \text{ then } 1 \text{ else } 1 * ((Y \text{ fact}) 0)$

$\rightarrow 1 * ((Y \text{ fact}) 0)$

$\rightarrow 1 * (\text{fact } (Y \text{ fact}) 0)$

$\rightarrow 1 * (\text{if } 0 = 0 \text{ then } 1 \text{ else } 0 * ((Y \text{ fact}) (-1)))$

$\rightarrow 1 * 1 \rightarrow 1$

Discussion

- ▶ Lambda calculus is Turing-complete
 - Most powerful language possible
 - Can represent pretty much anything in “real” language
 - Using clever encodings
- ▶ But programs would be
 - Pretty slow ($10000 + 1 \rightarrow$ thousands of function calls)
 - Pretty large ($10000 + 1 \rightarrow$ hundreds of lines of code)
 - Pretty hard to understand (recognize 10000 vs. 9999)
- ▶ In practice
 - We use richer, more expressive languages
 - That include built-in primitives

The Need For Types

- ▶ Consider the **untyped** lambda calculus
 - $\text{false} = \lambda x.\lambda y.y$
 - $0 = \lambda x.\lambda y.y$
- ▶ Since everything is encoded as a function...
 - We can easily misuse terms...
 - $\text{false } 0 \rightarrow \lambda y.y$
 - if 0 then ...
 - ...because everything evaluates to some function
- ▶ The same thing happens in assembly language
 - Everything is a machine word (a bunch of bits)
 - All operations take machine words to machine words

Simply-Typed Lambda Calculus

- ▶ $e ::= n \mid x \mid \lambda x:t.e \mid e e$
 - Added integers n as primitives
 - Need at least two distinct types (integer & function)...
 - ...to have type errors
 - Functions now include the type of their argument

Simply-Typed Lambda Calculus (cont.)

- ▶ $t ::= \text{int} \mid t \rightarrow t$
 - int is the type of integers
 - $t_1 \rightarrow t_2$ is the type of a function
 - That takes arguments of type t_1 and returns result of type t_2
 - t_1 is the **domain** and t_2 is the **range**
 - Notice this is a recursive definition
 - So we can give types to higher-order functions

Summary

- ▶ Lambda calculus shows issues with
 - Scoping
 - Higher-order functions
 - Types
- ▶ Useful for understanding how languages work