Due: Feb 21th at the start of class

Homework #2

CMSC351 - Spring 2013

PRINT Name

- Grades depend on neatness and clarity.
- Write your answers with enough detail about your approach and concepts used, so that the grader will be able to understand it easily. You should ALWAYS prove the correctness of your algorithms either directly or by referring to a proof in the book.
- Write your answers in the spaces provided. If needed, attach other pages.
- The grades would be out of 16. Four problems would be selected and everyone’s grade would be based only on those problems. You will also get 4 bonus points for trying to solve all problems.

1. Suppose you can only eat 1 or 2 apples every day. Let $F(0)=1$ and for $n \geq 1$ let $F(n)$ denote the different number of ways you can eat $n$ apples. Then $F(1)=1$ since your only option is to eat the apple on the first day. But $F(2)=2$ since you could either eat one apple each for two days, or eat both apples on the first day. Find a recurrence relation for $F(n)$. Show that $F(n) = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1})$ where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.
2. Solve the following recurrence relations using the Master Theorem, or just state that the Master Theorem does not apply.

(a) \( T(n) = T(n/2) + 2^n \)

(b) \( T(n) = 2^n \cdot T(n/2) + n^n \)

(c) \( T(n) = 3 \cdot T(n/3) + \left( \frac{n}{2} \right) \)

(d) \( T(n) = 4 \cdot T(n/2) + n^{2.5} \)
3. Function $T(n)$ is defined by the following recurrence relation.

$$T(n) = \begin{cases} 
8 & \text{if } 1 \leq n \leq 4, \\
4T\left(\left\lfloor \frac{n}{4} \right\rfloor \right) + 4n \log_2 n & \text{if } n > 4. 
\end{cases}$$

Prove $T(n) = O(n \log^2 n)$.

(Hint: use induction)
4. [Prob 3.22-a, Pg 58] Solve the following recurrence relation. It is sufficient to find the asymptotic behavior of $T(n)$. (Hint: Substitute another variable for $n$)

$$T(n) = \begin{cases} 
1 & \text{if } n = 2, \\
4T(\lfloor \sqrt{n} \rfloor) + 1 & \text{if } n > 2.
\end{cases}$$
5. Finding the majority: We have $n$ numbers such that one number has appeared at least $\left\lceil \frac{n}{2} \right\rceil + 1$ times. Design an algorithm that finds this number by at most $n$ comparisons between given numbers.
6. [Prob 5.22, Pg 116] Towers of Hanoi: There are \( n \) disks of different sizes arranged on a peg in decreasing order of sizes. There are two other empty pegs. The purpose of the puzzle is to move all the disks, one at a time, from the first peg to another peg in the following way. Disks are moved from the top of one peg to the top of another. A disk can be moved to a peg only if it is smaller than all other disks on that peg. In other words, the ordering of disks by decreasing sizes must be preserved at all times. The goal is to move all the disks in as few moves as possible.

a) Design an algorithm (by induction) to find a minimal sequence of moves that solves the towers of Hanoi problem for \( n \) disks.

b) How many moves are used in your algorithm? Construct a recurrence relation for the number of moves, and solve it.

c) Prove that the number of moves in part b is optimal; that is, prove that no algorithm can use fewer moves (use induction).
7. A board of size $2^n \times 2^n$ has an arbitrary square cut out of it. Prove the remaining board can be tiled using tiles of the following shape (rotation and reflection are allowed)? (Prove the statement for every $n>0$)

Here is the solution for a board of size $2^2 \times 2^2$. 

![Solution Diagram]