CMSC 631 – Program Analysis and Understanding
Spring 2013

Type Systems

The Need for a Type System

- Consider the (untyped) lambda calculus
  - false = $\lambda x.\lambda y.x$
  - 0 (Scott) = $\lambda x.\lambda y.x$
- Everything is encoded as a function
  - So we can easily misuse combinators
    - false 0   if 0 then ...   etc...
  - This is no better than assembly language!

What is a Type System?

- A *type system* is some mechanism for distinguishing good programs from bad
  - Good programs = well typed
  - Bad programs = ill typed or not typable

- Examples:
  - 0 + 1 // well typed
  - false 0 // ill-typed: can't apply a boolean
  - 1 + (if true then 0 else false) // ill-typed: can't add boolean to integer

A Definition of Type Systems

"A type system is a tractable syntactic method for proving the absence of certain program behaviors by classifying phrases according to the kinds of values they compute."

– Benjamin Pierce, *Types and Programming Languages*

Simply-Typed Lambda Calculus

- $e ::= n \mid x \mid \lambda x{:}t.e \mid e\ e$
  - Functions include the type of their argument
  - We don't really need this, but it will come in handy

- $t ::= \text{int} \mid t \rightarrow t$
  - $t1 \rightarrow t2$ is a the type of a function that, given an argument of type $t1$, returns a result of type $t2$
    - $t1$ is the *domain*, and $t2$ is the *range*

Type Judgments

- Our type system will prove *judgments* of the form
  - $A \vdash e : t$
  - "In type environment A, expression e has type t"

Type Environments

- A *type environment* is a map from variables to types (a kind of symbol table)
  - $\varnothing$ is the empty type environment
    - A closed term e is *well-typed* if $\varnothing \vdash e : t$ for some t
    - We'll abbreviate this as $\vdash e : t$
  - A, x:t is just like A, except x now has type t
    - The type of x in A, x:t is t
    - The type of $z \neq x$ in A, x:t in the type of z in A
- When we see a variable in a program, we look in the type environment to find its type

Type Rules

$$\frac{}{A \vdash n : \mathrm{int}}$$

$$\frac{x \in \mathrm{dom}(A)}{A \vdash x : A(x)}$$

$$\frac{A, x{:}t \vdash e : t'}{A \vdash \lambda x{:}t.e : t \to t'}$$

$$\frac{A \vdash e1 : t \to t' \quad A \vdash e2 : t}{A \vdash e1\ e2 : t'}$$

Example

$A = - : int \to int$

$$\frac{\dfrac{- \in dom(A)}{A \vdash - : int \to int} \qquad A \vdash 3 : int}{A \vdash -\ 3 : int}$$

Another Example

$A = + : \text{int} \to \text{int} \to \text{int}$

$B = A, x : \text{int}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{+ \in \text{dom}(B)}{B \vdash + :} \qquad \dfrac{x \in \text{dom}(B)}{B \vdash x : \text{i}}}{B \vdash +\ x : \text{int} \to \text{int}} \qquad B \vdash 3 : \text{int}}{B \vdash +\ x\ 3 : \text{int}}}{A \vdash (\lambda x{:}\text{int}.+\ x\ 3) : \text{int} \to \text{int}} \qquad A \vdash 4 : \text{int}}{A \vdash (\lambda x{:}\text{int}.+\ x\ 3)\ 4 : \text{int}}$$

We'd usually use infix x + 3

An Algorithm for Type Checking

- Our type rules are deterministic
  - For each syntactic form, only one possible rule

- They define a natural type checking algorithm
  - TypeCheck : type env × expression → type

TypeCheck(A, n) = int
TypeCheck(A, x) = if x in dom(A) then A(x) else fail
TypeCheck(A, λx:t.e) = t → (TypeCheck((A, x:t), e))
TypeCheck(A, e1 e2) =
    let t1 = TypeCheck(A, e1) in
    let t2 = TypeCheck(A, e2) in
        if dom(t1) = t2 then range(t1) else fail

Semantics

- Here is a small-step, call-by-value semantics
  - If an expression can't be evaluated any more and is not a value, then it is *stuck*

$$\frac{}{(\lambda x{:}t.e1)\ v2 \rightarrow e1[x \mapsto v2]} \qquad \frac{e1 \rightarrow e1'}{e1\ e2 \rightarrow e1'\ e2}$$

$$\frac{e2 \rightarrow e2'}{v1\ e2 \rightarrow v1\ e2'}$$

$e ::= v \mid x \mid e\ e$

$v ::= n \mid \lambda x{:}t.e$ *values* – not evaluated

Progress

- Suppose $\vdash e : t$. Then either e is a value, or there exists e' such that $e \to e'$
- Proof by induction on e
  - Base cases n, $\lambda x.e$ – these are values, so we're done
  - Base case x – can't happen (empty type environment)
  - Inductive case $e1\ e2$ – If $e1$ is not a value, then by induction we can evaluate it, so we're done, and similarly for $e2$. Otherwise both $e1$ and $e2$ are values. Inspection of the type rules shows that $e1$ must have a function type, and therefore must be a lambda since it's a value. Therefore we can make progress.

Preservation

- If $\vdash$ e : t and e → e′ then $\vdash$ e′ : t
- Proof by induction on e → e′
  - Induction (easier than the base case!). Expression e must have the form e1 e2.
  - Assume $\vdash$ e1 e2 : t and e1 e2 → e′. Then we have $\vdash$ e1 : t′ → t and $\vdash$ e2 : t′.
  - Then there are three cases.
    - If e1 → e1′, then by induction $\vdash$ e1’ : t′ → t, so e1′ e2 has type t
    - If reduction inside e2, similar

Preservation, cont'd

- Otherwise $(\lambda x{:}t'.e)\ v \rightarrow e[x \mapsto v]$. Then we have

$$\frac{x{:}\,t' \vdash e : t}{\vdash \lambda x{:}t'.e : t' \rightarrow t}$$

  - Thus we have
    - $x : t' \vdash e : t$
    - $\vdash v : t'$
  - Then by the substitution lemma (not shown) we have
    - $\vdash e[x \mapsto v] : t$
  - And so we have preservation

Substitution Lemma

- If $A \vdash v : t$ and $A, x{:}t \vdash e : t'$, then $A \vdash e[x \mapsto v] : t'$
- Proof: Induction on the structure of e
- For lazy semantics, we'd prove
  - If $A \vdash e1 : t$ and $A, x{:}t \vdash e : t'$, then $A \vdash e[x \mapsto e1] : t'$

Soundness

- So we have
  - Progress: Suppose $\vdash e : t$. Then either e is a value, or there exists e' such that $e \rightarrow e'$
  - Preservation: If $\vdash e : t$ and $e \rightarrow e'$ then $\vdash e' : t$
- Putting these together, we get soundness
  - If $\vdash e : t$ then either there exists a value v such that $e \rightarrow^* v$, or e diverges (doesn't terminate).
- What does this mean?
  - Evaluation getting stuck is bad, so
  - "Well-typed programs don't go wrong"

Product Types (Tuples)

$e ::= \ldots \mid (e, e) \mid \text{fst } e \mid \text{snd } e$

$$\frac{A \vdash e1 : t \qquad A \vdash e2 : t'}{A \vdash (e1,e2) : t \times t'}$$

$$\frac{A \vdash e : t \times t'}{A \vdash \text{fst } e : t} \qquad \frac{A \vdash e : t \times t'}{A \vdash \text{snd } e : t'}$$

- Or, maybe, just add functions
  - $\text{pair} : t \to t' \to t \times t'$
  - $\text{fst} : t \times t' \to t$
  - $\text{snd} : t \times t' \to t'$

Sum Types (Tagged Unions)

$e ::= \ldots \mid inL_{t2}\ e \mid inR_{t1}\ e$

$\mid (\text{case } e \text{ of } x1{:}t1 \rightarrow e1 \mid x2{:}t2 \rightarrow e2)$

$$\frac{A \vdash e : t1}{A \vdash inL_{t2}\ e : t1 + t2} \qquad \frac{A \vdash e : t2}{A \vdash inR_{t1}\ e : t1 + t2}$$

$$\frac{A \vdash e : t1 + t2 \qquad A, x1{:}t1 \vdash e1 : t \quad A, x2{:}t2 \vdash e2 : t}{A \vdash (\text{case } e \text{ of } x1{:}t1 \rightarrow e1 \mid x2{:}t2 \rightarrow e2) : t}$$

Self Application and Types

- Self application is not checkable in our system

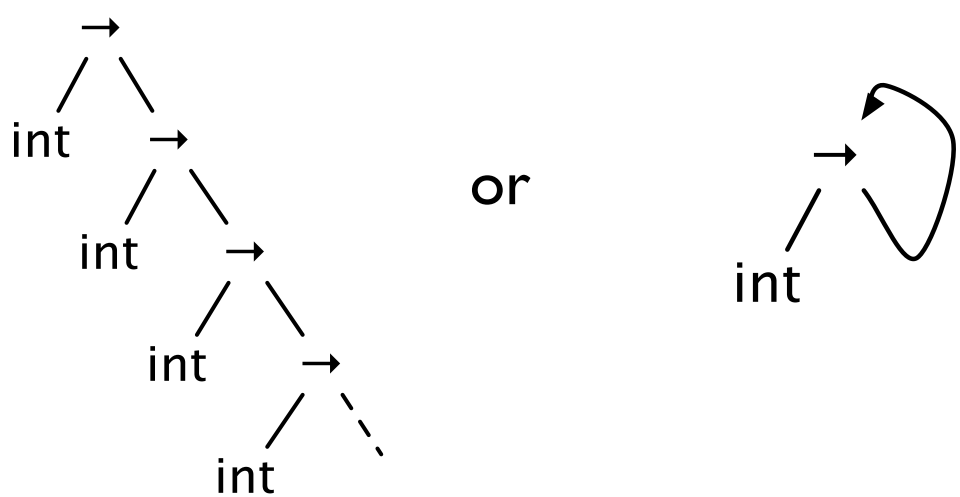
$$\dfrac{\dfrac{A, x{:}? \vdash x : t \to t' \qquad A, x{:}? \vdash x : t}{A, x{:}? \vdash x\ x : \ldots}}{A \vdash \lambda x{:}?.x\ x : \ldots}$$

  - It would require a type t such that $t = t \to t'$
    - (We'll see this next, but so far...)

- The simply-typed lambda calculus is *strongly normalizing*
  - Every program has a normal form
  - I.e., every program halts!

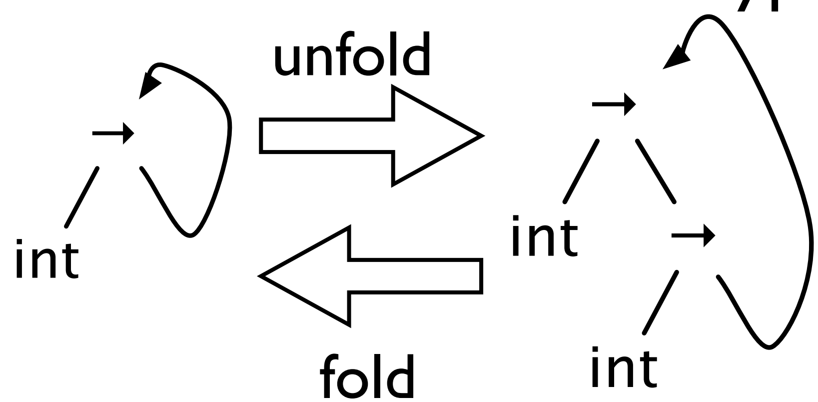
Recursive Types

- We can type self application if we have a type to represent the solution to equations like $t = t \to t'$
  - We define the type $\mu\alpha.t$ to be the solution to the (recursive) equation $\alpha = t$
  - Example: $\mu\alpha.\text{int}\to\alpha$



Folding and Unfolding

- We can check type equivalence with the previous definition (*equi-recursive types*)
  - Standard unification, omit occurs checks
- Alternative solution (*iso-recursive types*):
  - The programmer puts in explicit *fold* and *unfold* operations to expand/contract one "level" of the type trees
    - unfold $\mu\alpha.t = t[\alpha \mapsto \mu\alpha.t]$
    - fold $t[\alpha \mapsto \mu\alpha.t] = \mu\alpha.t$



Iso-recursive Types

- e ::= ... | fold e | unfold e

$$\frac{A \vdash e : t[\alpha \mapsto \mu\alpha.t]}{A \vdash \text{fold } e : \mu\alpha.t} \qquad \frac{A \vdash e : \mu\alpha.t}{A \vdash \text{unfold } e : t[\alpha \mapsto \mu\alpha.t]}$$

ML Datatypes

- Combines iso-recursive and sum types
  - Each occurrence of a type constructor when producing a value corresponds to occurrences of inL/inR and, when recursion is involved, fold
  - Each occurrence of a type constructor in a pattern match corresponds to a case and, when recursion is involved, (at least one) unfold

ML Datatypes Example

- type intlist = Int of int | Cons of int * intlist
  - Equivalent to $\mu\alpha.\text{int}+(\text{int} \times \alpha)$
- (Int 3) equivalent to
  - fold ($\text{inL}_{\text{int}\times\mu\beta.\text{int}+(\text{int}\times\beta)}$ 3)
- (Cons (2,(Int 3)) equivalent to
  - fold (inR_{int} (2, fold ($\text{inL}_{\text{int}\times\mu\beta.\text{int}+(\text{int}\times\beta)}$ 3)))
- match e with Int x -> e1 | Cons x -> e2 same as
  - case (unfold e)
    - $x{:}\text{int} \rightarrow e1$
    - | $x{:}\ \text{int}\times(\mu\beta.\text{int}+(\text{int}\times\beta)) \rightarrow e2$

Discussion

- In the pure lambda calculus, every term is typable with recursive types
  - (Pure = variables, functions, applications only)
- Most languages have some kind of "recursive" type
  - E.g., for data structures like lists, tree, etc.
- However, usually two recursive types that define the same structure but use a different name are considered different
  - E.g., struct foo { int x; struct foo *next; } is different from struct bar { int x; struct bar *next; }

Recap

- We've discussed simple types so far
  - Integers, functions, pairs, unions
  - Extensions for recursive types and updatable refs

- Type systems have nice properties
  - Type checking is straightforward (needs annotations)
  - Well typed programs don't go "wrong"
    - They don't get stuck in the operational semantics

- But...We can't type check all good programs

Up Next: Improving Types

- How can we build more flexible type systems?
  - More programs type check
  - Type checking is still tractable

- How can reduce the annotation burden?
  - Type inference

Parametric Polymorphism

- Observation: $\lambda x.x$ returns its argument exactly and places no constraints on the type of x
  - The identity function works for any argument type
- We can express this with *universal quantification:*
  - $\lambda x.x : \forall\alpha.\alpha \rightarrow \alpha$
  - For any type α, the identity function has type $\alpha \rightarrow \alpha$
  - This is also known as *parametric polymorphism*

System F: annotated polymorphism

- Let's extend our system as follows:
  - $t ::= \alpha \mid \text{int} \mid t \rightarrow t \mid \forall\alpha.t$
  - $e ::= n \mid x \mid \lambda x.e \mid e\ e \mid \Lambda\alpha.e \mid e\ [t]$
- That is, we add polymorphic types, and we add explicit *type abstraction (generalization)* …
  - Annotated code locations at which a value of polymorphic type is created
- … and *type application (instantiation)*
  - Explicitly annotated code locations at which a value of polymorphic type is used
- This system due to Girard, concurrently Reynolds

Defining Polymorphic Functions

- Polymorphic functions map types to terms
  - Normal functions map terms to terms
- Examples
  - $\Lambda\alpha.\lambda x{:}\alpha.x : \forall\alpha.\alpha\to\alpha$
  - $\Lambda\alpha.\Lambda\beta.\lambda x{:}\alpha.\lambda y{:}\beta.x : \forall\alpha.\forall\beta.\alpha\to\beta\to\alpha$
  - $\Lambda\alpha.\Lambda\beta.\lambda x{:}\alpha.\lambda y{:}\beta.y : \forall\alpha.\forall\beta.\alpha\to\beta\to\beta$

Instantiation

- When we use a parametric polymorphic type, we *apply* (or *instantiate*) it with a particular type
  - In System F this is done by hand:
  - $(\Lambda\alpha.\lambda x{:}\alpha.x)[t1] : t1 \to t1$
  - $(\Lambda\alpha.\lambda x{:}\alpha.x)[t2] : t2 \to t2$

- This is where the term *parametric* comes from
  - The type $\forall\alpha.\alpha\to\alpha$ is a "function" in the domain of types, and it is passed a parameter at instantiation time

Type Rules

$$\frac{A, \alpha \vdash e : t}{A \vdash \Lambda\alpha.e : \forall\alpha.t} \qquad \frac{A \vdash e : \forall\alpha.t}{A \vdash e[t'] : t[\alpha \mapsto t']}$$

- Notice that there are no constructs for manipulating values of polymorphic type
  - This justifies instantiation with *any* type—that's what the forall means!
- Note also that we are adding α to A; we could (should?) use this to ensure types are well-formed

Small-step Semantics Rules

$$\frac{}{(\Lambda\alpha.e)[t] \rightarrow e[\alpha \mapsto t]}\ \text{(type-app)} \qquad \frac{e \rightarrow e'}{e[t] \rightarrow e'[t]}\ \text{(tapp-cong)}$$

- We have to extend substitution to include types; that's up next … !

Free Variables, Again

- We're going to need to perform substitutions on quantified types
  - So just like with lambda calculus, we need to worry about free variables and capture-free substitution
- Define the free variables of a type
  - $FV(\alpha) = \{\alpha\}$
  - $FV(c) = \emptyset$
  - $FV(t \rightarrow t') = FV(t) \cup FV(t')$
  - $FV(\forall\alpha.t) = FV(t) - \{\alpha\}$

Substitution, Again

- Define $t[\alpha \mapsto u]$ as
  - $\alpha[\alpha \mapsto u] = u$
  - $\beta[\alpha \mapsto u] = \beta$ where $\beta \neq \alpha$
  - $(t \to t')[\alpha \mapsto u] = t[\alpha \mapsto u] \to t'[\alpha \mapsto u]$
  - $(\forall \beta.t)[\alpha \mapsto u] = \forall \beta.(t[\alpha \mapsto u])$ where $\beta \neq \alpha$ and $\beta \notin FV(u)$

- Define $e[\alpha \mapsto u]$ as
  - $(\lambda x{:}t.e)[\alpha \mapsto u] = \lambda x{:}t[\alpha \mapsto u].e[\alpha \mapsto u]$
  - $(\Lambda \beta.e)[\alpha \mapsto u] = \Lambda \beta.e[\alpha \mapsto u]$ where $\beta \neq \alpha$ and $\beta \notin FV(u)$
  - $(e1\ e2)[\alpha \mapsto u] = e1[\alpha \mapsto u]\ e2[\alpha \mapsto u]$
  - $x[\alpha \mapsto u] = x$ and $n[\alpha \mapsto u] = n$

An Imperative Language

$$
\begin{array}{llll}
e ::= x \mid \lambda x.e \mid e\ e & & & \\
 & \mid & \text{ref } e & \text{allocation} \\
 & \mid & !e & \text{dereference} \\
 & \mid & e := e & \text{assignment} \\
 & \mid & e;\ e & \text{sequencing}
\end{array}
$$

- Notice that this is not C
  - Variables cannot be updated; only references can
  - I.e., there are no l-values or r-values
- This is a language with *updatable references*

Examples

$!(\text{ref } 0)$

```
let x = ref 0 in
  x := !x + 1
```

```
let x = ref 0 in
  λy. x := !x + 1; !x
```

Type Checking Rules

- $t ::= \ldots \mid \text{ref } t$
  - Note: in ML this type is written t ref

$$\frac{A \vdash e : t}{A \vdash \text{ref } e : \text{ref } t}$$

$$\frac{A \vdash e : \text{ref } t}{A \vdash\ !e : t}$$

$$\frac{A \vdash e1 : \text{ref } t \quad A \vdash e2 : t}{A \vdash e1 := e2 : t}$$

Unit and the Unit Type

- Sometimes in imperative programs we write expressions that have some *side effect* but no interesting result
- To represent this directly, use *unit:*
  - e ::= ... | ()
  - t ::= ... | unit

$$\frac{}{A \vdash () : \mathrm{unit}} \qquad \frac{A \vdash e1 : \mathrm{ref}\ t \quad A \vdash e2 : t}{A \vdash e1 := e2 : \mathrm{unit}}$$

Operational Semantics

- Now we need to keep track of memory
  - State is a map from locations to values
  - Our redexes will be tuples ‹State, expression›
  - As a consequence, order of evaluation matters

- As before, evaluation will yield a fully-evaluated term, also called a *value*
  - v ::= x | λx.e
  - e ::= v | e e | ref e | !e | e := e

Operational Semantics (cont'd)

$$\frac{}{\langle S, (\lambda x.e1)\rangle \rightarrow \langle S, (\lambda x.e1)\rangle}$$

$$\frac{\langle S, e1\rangle \rightarrow \langle S', v1\rangle \quad \langle S', e2\rangle \rightarrow \langle S'', v2\rangle}{\langle S, e1; e2\rangle \rightarrow \langle S'', v2\rangle}$$

$$\frac{\langle S, e\rangle \rightarrow \langle S', v\rangle \qquad \text{loc fresh}}{\langle S, \text{ref } e\rangle \rightarrow \langle S[\text{loc}\mapsto v], \text{loc}\rangle}$$

Operational Semantics (cont'd)

$$\frac{\langle S, e\rangle \rightarrow \langle S', \mathit{loc}\rangle}{\langle S, !e\rangle \rightarrow \langle S', S'(\mathit{loc})\rangle}$$

$$\frac{\langle S, e1\rangle \rightarrow \langle S', \mathit{loc}\rangle \quad \langle S', e2\rangle \rightarrow \langle S'', v\rangle}{\langle S, e1 := e2\rangle \rightarrow \langle S''[\mathit{loc}\mapsto v], v\rangle}$$

$$\frac{\langle S, e1\rangle \rightarrow \langle S', \lambda x.e\rangle \quad \langle S', e2\rangle \rightarrow \langle S'', v\rangle \quad \langle S'', e[x\mapsto v]\rangle \rightarrow \langle S''', v'\rangle}{\langle S, e1\ e2\rangle \rightarrow \langle S''', v'\rangle}$$

Type Inference

- Let's reconsider the simply typed lambda calculus with integers
  - e ::= n | x | λx:t.e | e e
  - (No parametric polymorphism)

- *Type inference*: Given a bare term (with no type annotations), can we reconstruct a valid typing for it, or show that it has no valid typing?

Type Language

- Problem: Consider the rule for functions

$$\frac{A, x{:}t \vdash e : t'}{A \vdash \lambda x{:}t.e : t \rightarrow t'}$$

- Without type annotations, where do we get t?
  - We'll use *type variables* to stand for as-yet-unknown types
    - $t ::= \alpha \mid \text{int} \mid t \rightarrow t$
  - We'll generate *equality constraints* $t = t$ among the types and type variables
    - And then we'll solve the constraints to compute a typing

Type Inference Rules

$$\frac{}{A \vdash n : int}$$

$$\frac{x \in dom(A)}{A \vdash x : A(x)}$$

$$\frac{A, x{:}\alpha \vdash e : t' \qquad \alpha \text{ fresh}}{A \vdash \lambda x.e : \alpha \rightarrow t'}$$

$$\frac{A \vdash e1 : t1 \qquad A \vdash e2 : t2 \qquad \boxed{t1 = t2 \rightarrow \beta} \qquad \beta \text{ fresh}}{A \vdash e1\ e2 : \beta}$$

"Generated" constraint

Example

$$\dfrac{\dfrac{A, x{:}\alpha \vdash x{:}\alpha}{A \vdash (\lambda x.x) : \alpha \to \alpha} \qquad A \vdash 3 : \mathrm{int} \qquad \alpha \to \alpha = \mathrm{int} \to \beta}{A \vdash (\lambda x.x)\ 3 : \beta}$$

- We collect all constraints appearing in the derivation into some set C to be solved
- Here, C contains just $\alpha \to \alpha = \mathrm{int} \to \beta$
  - Solution: $\alpha = \mathrm{int} = \beta$
- Thus this program is typable, and we can derive a typing by replacing α and β by int in the proof

Solving Equality Constraints

- We can solve the equality constraints using the following rewrite rules, which reduce a larger set of constraints to a smaller set
  - $C \cup \{int=int\} \Rightarrow C$
  - $C \cup \{\alpha=t\} \Rightarrow C[\alpha \mapsto t]$
  - $C \cup \{t=\alpha\} \Rightarrow C[\alpha \mapsto t]$
  - $C \cup \{t1 \rightarrow t2=t1' \rightarrow t2'\} \Rightarrow C \cup \{t1=t1'\} \cup \{t2=t2'\}$
  - $C \cup \{int=t1 \rightarrow t2\} \Rightarrow$ unsatisfiable
  - $C \cup \{t1 \rightarrow t2=int\} \Rightarrow$ unsatisfiable

Termination

- We can prove that the constraint solving algorithm terminates.
- For each rewriting rule, either
  - We reduce the size of the constraint set
  - We reduce the number of "arrow" constructors in the constraint set
- As a result, the constraint always gets "smaller" and eventually becomes empty
  - A similar argument is made for strong normalization in the simply-typed lambda calculus

Occurs Check

- We don't have recursive types, so we shouldn't infer them
- So in the operation $C[\alpha \mapsto t]$, require that $\alpha \notin FV(t)$
  - (Except if t = a, in which case there's no recursion in the types, so unification should succeed)
- In practice, it may better to allow $\alpha \in FV(t)$ and do the occurs check at the end
  - But that can be awkward to implement

Unifying a Variable and a Type

- Computing $C[\alpha \mapsto t]$ by substitution is inefficient

- Instead, use a union-find data structure to represent equal types
  - The terms are in a union-find forest
  - When a variable and a term are equated, we union them so they have the same ECR (equivalence class representative)
    - Want the ECR to be the concrete type with which variables have been unified, if one exists. Can read off solution by reading the ECR of each set.

Example

$\alpha = int \rightarrow \beta$

$\gamma = int \rightarrow int$

$\alpha = \gamma$

Unification

- The process of finding a solution to a set of equality constraints is called *unification*
  - Original algorithm due to Robinson
    - But his algorithm was inefficient
  - Often written out in different form
    - See Algorithm W
  - Constraints usually solved on-line
    - As type inference rules applied

Discussion

- The algorithm we've given finds the *most general type* of a term
  - Any other valid type is "more specific," e.g.,
    - $\lambda x.x : int \rightarrow int$
  - Formally, any other valid type can be gotten from the most general type by applying a substitution to the type variables

- This is still a *monomorphic* type system
  - α stands for "some particular type, but it doesn't matter exactly which type it is"

Inference for Polymorphism

- We would like to have the power of System F, and the ease of use of type inference
  - In short: given an untyped lambda calculus term, can we discover the annotations necessary for typing the term in System F, if such a typing is possible?
  - Unfortunately, no. This problem has been shown to be undecidable.
- Can we at least perform some type inference for parametric polymorphism?
  - Yes. A sweet spot was found by Hindley and Milner
  - But first, let's consider the general case …

Attempting Type Inference

- Let's extend simply-typed calculus as follows:
  - $t ::= \alpha \mid \text{int} \mid t \to t \mid \forall\alpha.t$
  - $e ::= n \mid x \mid \lambda x.e \mid e\ e$
- Type inference will automatically infer where to generalize a term, to introduce polymorphic types, and where to instantiate them

Instantiation

$$\frac{A \vdash e : \forall\alpha.t}{A \vdash e : t[\alpha \mapsto t']}$$

- This rule is exacty the same as System F, but we just "magically" pick which t' to instantiate with
  - You're surely wondering about algorithmics. We'll get to that …

Generalization

- Question: When is it safe to generalize (quantify) a type variable α in the type of expression e?

- Answer: Whenever we can redo the typing proof for e, choosing α to be anything we want, and still have a valid typing proof.

Examples

$$\frac{A, x{:}\alpha \vdash e : \alpha}{A \vdash \lambda x.x : \alpha \to \alpha}$$

$$\frac{A, x{:}int \vdash x : int}{A \vdash \lambda x.x : int \to int}$$

$$\frac{A, x{:}(i \to i) \vdash x : (i \to i)}{A \vdash \lambda x.x : (i \to i) \to (i \to i)}$$

- The choice of the type of x is purely local to type checking λx.x
  - There is no interaction with the outside environment
  - Thus we can generalize the type of x

Examples (cont'd)

$$\frac{A, x{:}\mathrm{int} \vdash x : \mathrm{int}}{A \vdash \lambda x.x{+}3 : \mathrm{int} \rightarrow \mathrm{int}}$$

- The function restricts the type of x, so we cannot introduce a type variable
  - Thus we cannot generalize the type of x
  - We can only generalize when the function doesn't "look at" its parameter

Examples (cont'd)

$$\frac{A, y{:}\alpha, x{:}\alpha \vdash \text{if } p \text{ then } x \text{ else } y : \alpha}{A, y{:}\alpha \vdash \lambda x.\text{if } p \text{ then } x \text{ else } y :}$$

$$\Downarrow$$

$$\xcancel{\frac{A, y{:}\alpha, x{:}\text{int} \vdash \text{if } p \text{ then } x \text{ else } y : \text{int}}{A, y{:}\alpha \vdash \lambda x.\text{if } p \text{ then } x \text{ else } y : \text{int} \rightarrow \text{int}}}$$

- The choice of the type of x depends on the type environment
  - In the first derivation, x and y have the same type; if we generalize the type of x, they could have different types
  - Thus we cannot generalize the type of x

Generalization Rule

$$\frac{A \vdash e : t \qquad \alpha \notin FV(A)}{A \vdash e : \forall\alpha.t}$$

- We can generalize any type variable that is unconstrained by the environment
  - Warning: This won't quite work with refs

Another Justification

- Suppose we have
  - $A \vdash e : t$ and $\alpha \notin FV(A)$
- Then let u be any type. By induction, can show
  - $A[\alpha \mapsto u] \vdash e : t[\alpha \mapsto u]$
  - But then since $\alpha \notin FV(A)$, that's equivalent to
  - $A \vdash e : t[\alpha \mapsto u]$

Polymorphic Type Inference

- We'd like to extend our algorithm to polymorphic type inference
  - Performance generalization and instantiation automatically (and deterministically)

- Major problem: Our system for polymorphism is too expressive

Hindley-Milner Polymorphism

- Restrict polymorphism to only the "top level"
  - Introduce polymorphism at let
  - Fully instantiate at use of a polymorphic type
- Here is our new language
  - $e ::= n \mid x \mid \lambda x.e \mid e\ e \mid \text{let } x = e \text{ in } e$
  - $t ::= \alpha \mid \text{int} \mid t \to t$
  - $s ::= t \mid \forall \alpha.s$
    - These are *type schemes*
  - $A ::= \varnothing \mid A, x{:}s$
  - Notice that, according to the prior instantiation rule, we won't instantiate α with a scheme s, only a type t

Old Type Inference Rules

$$\frac{}{A \vdash n : int}$$

$$\frac{A, x{:}\alpha \vdash e : t' \quad \alpha \text{ fresh}}{A \vdash \lambda x.e : \alpha \to t'}$$

$$\frac{A \vdash e1 : t1 \quad A \vdash e2 : t2 \qquad t1 = t2 \to \beta \quad \beta \text{ fresh}}{A \vdash e1\ e2 : \beta}$$

New Type Inference Rules

- At let, generalize over all possible variables

$$\frac{A \vdash e1 : t1 \quad A, x{:}\forall\vec{\alpha}.t1 \vdash e2 : t2 \quad \vec{\alpha} = FV(t1) - FV(A)}{A \vdash \text{let } x = e1 \text{ in } e2 : t2}$$

- At variable uses, instantiate to all fresh types

$$\frac{A(x) = \forall\vec{\alpha}.t \quad \vec{\beta} \text{ fresh}}{A \vdash x : t[\vec{\alpha} \mapsto \vec{\beta}]}$$

  - Here the $\vec{\alpha}$ denotes a list of type variables

Algorithm W

- A type inference algorithm that explicitly solves the equality constraints on-line

- Instead of implicit global substitution (like we used before), threads the substitution through the inference

- In practice, use previous algorithm, plus generalize at let and instantiate at variable uses.
  - Solve for the type of e1, generalize it, then instantiate its solution when doing inference on e2

Example

- Parametric polymorphic type inference

```
let x = λx.x in       // x : ∀α.α→α

   x 3;               // x : β→β,   β=int
   x (λy.y)           // x : γ→γ,   γ=δ→δ
```

- This would be untypable in a monomorphic type system

Kinds of Polymorphism

- We've just seen parametric polymorphism
  - System F and Hindley-Milner style polymorphism
- Another popular form is subtype polymorphism
  - As in OO programming
  - These two can be combined (e.g., Java Generics)
- Some languages also have *ad-hoc polymorphism*
  - E.g., + operator that works on ints and floats
  - E.g., overloading in Java

Polymorphism and References

- Suppose we want polymorphism in our imperative language
  - e ::= x | n | λx.e | e e | ref e | !e | e := e
  - s ::= t | ∀α.s
  - t ::= α | int | t → t | ref t
- What if we try our standard rule?

$$\frac{A \vdash e1 : t1 \quad A, x{:}\forall\vec{\alpha}.t1 \vdash e2 : t2 \quad \vec{\alpha} = FV(t1) - FV(A)}{A \vdash \text{let } x = e1 \text{ in } e2 : t2}$$

Naive Generalization is Unsound

- Example (due to Tofte)

  let r = ref (λx.x) in   // r : $\forall\alpha$.ref ($\alpha \to \alpha$)

    r := λx.x+1;   // checks; use r at ref (int $\to$ int)

    (!r) true   // oops! checks; use r at ref(bool $\to$ bool)

- α should not be generalized, because later uses of r may place constraints on it

- Nobody realized there was a problem for a long time

Solution: The Value Restriction

- Only allow *values* to be generalized
  - $v ::= x \mid n \mid \lambda x.e$
  - $e ::= v \mid e\ e \mid \text{ref}\ e \mid !e \mid e := e$

$$\frac{A \vdash v : t1 \qquad A, x{:}\forall\alpha.t \vdash e2 : t2 \qquad \alpha = FV(t) - FV(A)}{A \vdash \text{let}\ x = v\ \text{in}\ e2 : t2}$$

  - Intuition: Values cannot later be updated
  - This solution due to Wright and Felleisen
    - Tofte found a much more complicated solution

Benefits of Type Inference

- Handles higher-order functions
- Handles data structures smoothly
- Works in infinite domains
  - Set of types is unlimited
- No forward/backward distinction
- Polymorphism provides context-sensitivity

Drawbacks to Type Inference

- Flow-insensitive
  - Types are the same at all program points
  - May produce coarse results
  - Type inference failure can be hard to understand
- Polymorphic type inference may not scale
  - Exponential in worst case
  - Seems fine in practice (witness ML)