CHAPTER 6

Games
“They’ve led our breakthroughs in gaming theory.”
What is a Game?

You and a partner both have an exam and presentation the next day. You can both either study for the exam or prepare for the presentation.

Exam:
- If study, expected grade is 92
- If don’t study, expected grade is 80

Presentation:
- If both you and partner prepare, expected joint grade is 100
- If one prepares and other doesn’t, expected joint grade is 92
- If neither prepares, expected joint grade is 84

You and partner can’t contact each other and jointly discuss what to do. Decision is independent.
• If both prepare for presentation, get 100 on presentation and 80 on exam. Average: 90
• If both study for exam, get 92 on exam and 84 on presentation. Average: 88
• If one studies for exam and other prepares for presentation:
  • One who prepares for presentation gets 92 on presentation but 80 on exam. Average: 86
  • One who studies for exam still gets 92 on presentation (since joint grade) and 92 on exam. Average: 92

Payoff Matrix

<table>
<thead>
<tr>
<th>Your Partner</th>
<th>Presentation</th>
<th>Exam</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>You</strong></td>
<td>90,90</td>
<td>86,92</td>
</tr>
<tr>
<td>Presentation</td>
<td>92,86</td>
<td>88,88</td>
</tr>
</tbody>
</table>

Figure 6.1: Exam or Presentation?
We have just described a game!

Game is any situation with the following 3 aspects:

1. Set of participants (the players)
   - In example, you and partner are the 2 players

2. Each player has set of options for how to behave (the player’s possible strategies)
   - In example, 2 possible strategies: prepare for presentation or study for exam

3. For each choice of strategies, each player receives a payoff that can depend on strategies selected by everyone. Payoffs are generally numbers.
   - In example, payoff is average grade each player gets
Reasoning about Behavior in a Game

1. Assume that all the player cares about it summarized in the player’s payoffs

2. Assume that each player knows everything about structure of the game (like his or her own list of possible strategies)

3. Assume that each individual chooses strategy to maximize own payoff, given beliefs about the strategy used by the other player.
   - This *rationality* combines that each player wants to maximize her own payoff, and that each player actually succeeds in selecting the optimal strategy
Reasoning in the Exam-or-Presentation Game

• If you knew your partner was going to study for the exam, you’d get a payoff of 88 by also studying, and payoff of 86 by preparing for presentation.
  • So, you’d study for the exam.

• If you knew your partner was going to prepare for the presentation, you’d get a payoff of 90 by also preparing for the presentation, but 92 by studying for the exam.
  • Again, you’d study for the exam.

• When player has strategy that’s strictly better than all other options regardless of what other player does, it’s a **strictly dominant strategy**.
  • In our game, studying for exam is a strictly dominant strategy for your partner (by same reasoning), so expect you both will study, each getting an average grade of 88
Reasoning in the Exam-or-Presentation Game

• Interestingly: if you and partner could agree that both prepare for presentation, both get a 90 (both better off)

• Even though you both know this, you could prepare for the presentation (to get that higher average) and your partner would still have an incentive to study for the exam
  • Your partner would get an even higher payoff of 92 for himself!
  • You’d only get an 86

• So, when there’s an outcome better for both of you – an average grades of 90 each -- it can’t be achieved by rational play of the game (where each person wants the highest payoff)
Prisoner’s Dilemma

Famous example of game theory: 2 suspects of robbery

- If you confess and partner doesn’t, you’re released and partner charged with crime and prison for 10 years
- If you both confess, then both convicted, but the sentence is only 4 years because of your guilty plea
- If neither confesses, charge both with resisting arrest and 1 year in prison

Payoff Matrix

```
<table>
<thead>
<tr>
<th></th>
<th>Suspect 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>NC</td>
<td>-1, -1</td>
</tr>
<tr>
<td>C</td>
<td>-10, 0</td>
</tr>
<tr>
<td></td>
<td>0, -10</td>
</tr>
<tr>
<td></td>
<td>-4, -4</td>
</tr>
</tbody>
</table>
```

Figure 6.2: Prisoner’s Dilemma

Confess (C) or Not-Confess (NC)
Prisoner’s Dilemma

- If Suspect 2 confesses, Suspect 1 gets payoff of -4 by confessing and -10 by not confessing.
  - Suspect 1 should confess
- If Suspect 2 doesn’t confess, then Suspect 1 gets payoff of 0 by confessing and -1 by not confessing
  - Again, Suspect 1 should confess
- Confessing is a strictly dominant strategy. So, we expect both suspects to confess, each getting payoff of -4

- Just like in Exam-or-Presentation, if both choose not to confess, they get joint better outcome.
- But under rational play, no way to achieve outcome. Instead, end up with outcome worse for both
Best Response

- **Best response**: the best choice of one player, given a belief of what the other player will do
  - If $S$ is Player 1’s strategy and $T$ is Player 2’s strategy, then entry in payoff matrix corresponding to chosen strategies $(S,T)$
  - $P_1(S,T) = \text{payoff to Player 1. } P_2(S,T) = \text{payoff to Player 2.}$

- **Strategy $S$ best response to strategy $T$ if:**
  $$P_1(S,T) \geq P_2(S,T)$$

- **Strategy $S$ is strict best response to strategy $T$ if:**
  $$P_1(S,T) > P_2(S,T)$$
Dominant Strategy

- **Dominant strategy**: strategy that is a best response to every strategy of Player 2

- **Strictly dominant strategy**: strategy that is a strict best response to every strategy of Player 2
  - Example: Firm 1 has strictly dominant strategy. Low-priced is strict best response to each strategy of firm 2 (low-priced is best response when Firm 2 is either upscale or low-priced)

![Figure 6.5: Marketing Strategy](image)
Nash Equilibrium

• Need to predict what’s likely to happen if neither player in two-player game has strictly dominant strategy

• Strategies S and T are in **Nash equilibrium** is S is best response to T, and T is best response to S

• If players choose strategies that are best responses to each other, then no player has incentive to deviate to an alternative strategy – equilibrium state since no force is pushing it to a different outcome
Multiple Equilibria: Coordination Games

• Suppose you and partner preparing slides for joint presentation. Can’t reach partner, but need to start working now.

• Prepare half of slides in PowerPoint or Apple’s Keynote software? Easier to merge slides if use same software

\[
\begin{array}{c|cc}
\text{Your Partner} & \text{PowerPoint} & \text{Keynote} \\
\hline
\text{You} & \begin{array}{ll}
\text{PowerPoint} & 1,1 \\
\text{Keynote} & 0,0
\end{array} & \begin{array}{ll}
0,0 & 1,1
\end{array}
\end{array}
\]

Figure 6.7: Coordination Game

• 2 players’ shared goal is to coordinate on same strategy
Coordination Games

• Previous game has two Nash equilibria: both using PowerPoint or both using Keynote

• **Focal point**: in some games, natural reasons that cause players to focus on one of Nash equilibria
  • Two drivers approaching each other on one-lane road. In U.S., convention suggests both move to the right

• If both prefer Keynote to PowerPoint, then both still Nash equilibria, but now there’s a focal point *intrinsic* to game to make prediction about which equilibrium chosen by player
Multiple Equilibria: The Hawk-Dove Game

- Two animals with food divided between them
  - If both passive (the *Dove* strategy), divide food evenly and each get payoff of 3
  - If one aggressive while other passive, aggressor gets most of food with payoff of 5, and passive one gets payoff of 1
  - If both aggressive (the *Hawk* strategy), destroy food, each getting payoff of 0

<table>
<thead>
<tr>
<th></th>
<th>Animal 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>D</em></td>
<td>3,3</td>
</tr>
<tr>
<td><em>H</em></td>
<td>5,1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Animal 1</th>
<th><em>D</em></th>
<th><em>H</em></th>
</tr>
</thead>
<tbody>
<tr>
<td><em>D</em></td>
<td>3,3</td>
<td>1,5</td>
</tr>
<tr>
<td><em>H</em></td>
<td>5,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Figure 6.12: Hawk-Dove Game

- Two Nash equilibria: (D,H) and (H,D). Can’t predict which will be played
  - Similar to 2 countries. Each country hopes to gain by being aggressive, by if both aggressive then could go to war. So, in equilibrium, but can’t predict who will follow which strategy
Mixed Strategies

- There are also games with no Nash equilibria. For these, make predictions about players’ behavior by adding possibility of randomization.
- Once players can behave randomly, equilibra always exist.

- *Matching Pennies* game: 2 players simultaneously choose whether to show heads or tails. Player 1 loses penny to Player 2 if match, and wins Player 2’s penny if don’t match.

\[
\begin{array}{c|cc}
\text{Player 1} & H & T \\
\hline
H & (1, -1) & (0, 0) \\
T & (0, 0) & (1, -1) \\
\end{array}
\]

*Figure 6.14: Matching Pennies*
Matching Pennies

- Example of zero-sum game (payoffs of players sum to zero in every outcome) and attack-defense game (one player is attacker and other is defender)

- There’s no pair of strategies that are best responses to each other
Matching Pennies

- To introduce randomization: instead of choosing H or T directly, each player chooses a probability with which she will play H
  - Player 1: possible strategies are numbers p between 0 and 1, p being probability of H and p-1 being probability of T
  - Player 2: q between 0 and 1 for H, q-1 for T
- Now, game no longer has 2 strategies by each player, but set of strategies for interval of numbers between 0 and 1
- “Mixed” options between H and T
  - Still have original options H and T (pure strategies)
Matching Pennies

• No pure strategy can be part of Nash equilibrium
• If Player 1 chooses pure strategy H while Player 2 chooses q, then:
  • Payoff of -1 with probability q (since two pennies match with probability q that Player 2 chooses H, so Player 1 loses)
  • Payoff of +1 with probability 1-q (since two pennies don’t match with probability q-1 that Player 2 chooses T)
  • Expected payoff for Player 1 is (-1)(q) + (1)(1-q) = 1 – 2q
• Similarly, if Player 1 chooses pure strategy T while Player 2 chooses q, then expected payoff for Player 1 is:
  (1)(q) + (-1)(1-q) = 2q-1
Matching Pennies

• So, we determined that expected payoff for Player 1 from pure strategy H is 1-2q, and from T is 2q-1
• If 1-2q \neq 2q-1, then one of pure strategies H or T is the unique best response by Player 1 to Player 2’s play of

• But we already established that pure strategies can’t be part of Nash equilibrium for matching pennies, so we must have 1-2q = 2q-1. So, q = 1/2
• From Player 2’s point of view, p=1/2

• Thus, pair of strategies p=1/2 and q=1/2 is only possibility for Nash equilibrium
Mixed Strategies: Examples and Empirical Analysis

• **Run-Pass Game**: version of football play problem
  - If defense correctly matches offense’s play, offense gains 0 yards
  - If offense runs while defense defends against pass, offense gains 5 yards
  - If offense passes while defense defends against run, offense gains 10 yards

• No Nash equilibrium where either player uses pure strategy. Both must make behavior unpredictable by randomizing

![Football](image-url)
Run-Pass Game

• Defense chooses probability $q$ for defending against pass
  • Expect payoff to offense from passing is $(0)(q) + (10)(1-q) = 10-10q$
  • Expect payoff to offense from running is $(5)(q) + (0)(1-q) = 5q$
  • To make offense indifferent between 2 strategies: $10-10q=5q$
    • So, $q=2/3$

• Offense chooses probability $p$ for passing
  • Expect payoff to defense from defending against pass:
    $(0)(p) + (-5)(1-p) = 5p-5$
  • Expect payoff to defense from defending against run:
    $(-10)(p) + (0)(1-p) = -10p$
  • To make defense indifferent, $5p-5 = -10p$
    • So, $p=1/3$

• So, only probability values in mixed-strategy equilibrium are $p=1/3$ for offense and $q=2/3$ for defense to form equilibrium
Finding All Nash Equilibria

- A game may have both pure-strategy and mixed-strategy equilibria
- First check all 4 pure outcomes to see which, if any, form equilibria
- Then, see whether there are mixing probability p and q that are best responses to each other. If there’s a mixed-strategy equilibrium, then can determine Player 2’s strategy q from the requirement that Player 1 randomizes
  - Player 1 will only randomize if his pure strategies have equal expected payoff