Announcements

• Homework #2 has been posted
Writing Proofs

A good proof should have:

– a statement of what is to be proven
– "Proof:" to indicate where the proof starts
– a clear indication of flow
– a clear indication of the reason for each step
– careful notation, completeness and order
– a clear indication of the conclusion
Statement of Claims

The following are equivalent:

• Every even number (greater than 3) is the sum of two primes.
• For all $n \in \mathbb{N}^{\text{Even}}$: If $n > 3$, then $n$ is the sum of two primes.
• $(\forall n \in \mathbb{N}^{\text{Even}})[n > 3 \rightarrow (\exists a, b \in \mathbb{N}^{\text{Prime}})[n = a + b]]$
• $(\forall n \in \mathbb{N})[(\exists k \in \mathbb{N})[n = 2k] \land n > 3 \rightarrow (\exists a, b \in \mathbb{N})[a > 1 \land (\forall c, d \in \mathbb{N})[cd = a \rightarrow c = 1 \lor d = 1] \land b > 1 \land (\forall e, f \in \mathbb{N})[ef = b \rightarrow e = 1 \lor f = 1] \land n = a + b]]$

Which of these would you use to state your claim?
Constructive Proofs of Existence

• Claim: $(\exists a, b \in \mathbb{N})[a^b = b^a \land a \neq b]$

• Claim: There exist three natural numbers, $a$, $b$, and $c$ (all distinct) such that $a^2 + b^2 = c^2$

• Claim: 23 can be written as the sum of 9 cubes (of non-negative integers).

• Claim: There is a number that can be written as the sum of two cubes (of positive integers) in two different ways.

• Talk about “Taxicab” numbers, and “non-constructive” proofs of existence.
Proofs by Exhaustion/Cases

• Claim: \( \forall n \in \{1, 2, 3, 4\} \ [(n + 1)^3 \geq 3^n] \).

• Claim: There are no integer solutions to the equation \( a^2 + b^2 = 7 \)

• Claim: 23 cannot be written as the sum of 8 cubes (of non-negative integers).

• Mention proof of four-color problem
Applying Universal Generalization

- The most common technique for proving *universally quantified* statements.
- If you’re not sure how to start – try this!

Claim: $(\forall x \in D)[P(x)]$

Proof:

Let $d \in D$, *arbitrarily chosen*.

... 

$P(d)$

Since $d$ was chosen arbitrarily, $P(x)$ holds for all $x \in D$. 
Example of Proving a Universal Statement

Claim: \((\forall n \in \mathbb{N}^{\text{Even}})[n^2 \text{ is even}]\)
Proof Example: Rigid Style

**Claim:** \((\forall n \in \mathbb{N}^{\text{Even}}) [n^2 \text{ is even}]\)

**Proof:**

1. Let \(a \in \mathbb{N}^{\text{Even}}\), selected arbitrarily
2. \(a = 2k\), for some \(k \in \mathbb{N}\)  \[Defn \text{ of “even”}]  
3. \(a^2 = (2k)(2k)\)
4. \(a^2 = 2(2k^2)\)
5. \((\text{Note that } 2k^2 \in \mathbb{N})\)  \[Since \(\mathbb{N}\) is closed under multiplication\]  
6. \(a^2 \text{ is even} \)  \[Defn \text{ of “even”; using (4), (5)}\]  
7. Since \(a\) was chosen arbitrarily, \((\forall n \in \mathbb{N}^{\text{Even}}) [n^2 \text{ is even}]\)
**Proof Example: Flowing Style**

**Claim:** The square of any even natural number is even.

**Proof:**

Let \( a \in \mathbb{N}_{\text{Even}} \), selected arbitrarily. Since \( a \) is even, \( a = 2k \) for some \( k \in \mathbb{N} \). Squaring both sides, we get \( a^2 = (2k)^2 = 2(2k^2) \). Noting that \( 2k^2 \in \mathbb{N} \) (\( \mathbb{N} \) is closed under multiplication), we see that \( a^2 \) is equal to twice a natural number, hence \( a^2 \) is even. Since \( a \) was selected arbitrarily, the proposition holds for any even number.

**Questions:**

- Which style do you think is easier to understand?
- Which style is easier to write without making mistakes?
- Can we use a style that is somewhere between these two?
More Examples

• Claim: The product of two odd integers is odd.
• Claim: $\mathbb{Q}$ is closed under multiplication. (Assuming we know that $\mathbb{Z}$ is closed under multiplication.)
• Claim: $(\forall n \in \mathbb{N}^{>0}), n^2 + 3n + 2$ is composite.