Announcements

• Homework #3 has been posted
One More Basic Example

- Claim: \( \mathbb{Q} \) is dense. (Assuming we already know about the closure of \( \mathbb{Q} \).)
More with Cases

- Claim: For all integers, $x$ and $y$, $|x/y| = |x|/|y|$
- Claim: For all $n \in \mathbb{N}$, $3n^2 + n + 14$ is even.
Fun Proof

An existence proof that is as close to “constructive” as you can get without actually being constructive...

- Claim: There are two irrational numbers, $a$ and $b$, such that $a^b$ is rational.
Suppose \( ab = c \), where \( a, b, c \in \mathbb{Z} \) (with \( b \neq 0 \))

- We use the following notation to express “\( b \) divides \( c \)”
  \[
  b \mid c
  \]

- In proofs, we frequently use the following interchangeably:
  \[
  b \mid c \text{ is the same as } (\exists a \in \mathbb{Z})[c = ab]
  \]
Implications

Below are outlines of the standard technique for proving implications:

**Claim:** If P then Q.

**Proof:**
- Assume P.
- ... 
- Q.
- Therefore, P → Q.

**Claim:** (∀x ∈ D) [If P(x) then Q(x)].

**Proof:**
- Let d ∈ D, selected arbitrarily.
- Assume P(d) holds.
- ... 
- Q(d).
- Therefore, P(d) → Q(d).
- Since d was selected arbitrarily,
  (∀x ∈ D) [P(x) → Q(x)].
Examples with implications

• Claim: \( \forall x, y, z \in \mathbb{N} \): If \( x \mid y \), and \( y \mid z \), then \( x \mid z \).

• Claim: \( \forall x, y \in \mathbb{R} \), if \( x + y = 7 \) and \( xy = 10 \), then \( x^2 + y^2 = 29 \).
Proof by Contrapositive

Sometimes implications are easier to prove this way:

Claim: If P then Q.

Proof:
Assume ~Q.
...
~P.
Therefore, P→Q.
Examples using Contrapositive

• \((\forall n \in \mathbb{N})\) If \(3n + 2\) is odd then \(n\) is odd.

• \((\forall n, a, b \in \mathbb{R}^+)\) If \(n = ab\) then \(a \leq \sqrt{n}\) or \(b \leq \sqrt{n}\).
Proofs of Equivalence ("If and only if")

Two techniques:

Claim: \( P \iff Q. \)

Proof:
\[
\begin{align*}
P & \iff S_1 \\
\iff S_2 \\
\iff S_3 \\
\ldots \\
\iff Q
\end{align*}
\]

• Doesn’t always work
• Easy to make mistakes
• Maybe less writing

Claim: \( P \iff Q. \)

Proof:

Part I. [Show \( P \implies Q \)]
\[
\]
Part II. [Show \( Q \implies P \)]
\[
\]
• Works more often
• Less error prone
• Probably more writing
Be careful!

Critique this “proof”.

**Warning:** This proof is invalid!

**Claim:** \((\forall n \in \mathbb{N})[n \text{ is odd} \iff n^2 \text{ is odd}]\)

**Proof:**

Let \(a \in \mathbb{N}\), selected arbitrarily.

- \(a \text{ is odd} \iff a = 2k + 1 \text{ (some } k \in \mathbb{N})\)
  - \(\iff a^2 = (2k + 1)(2k + 1)\)
  - \(\iff a^2 = 4k^2 + 4k + 1\)
  - \(\iff a^2 = 2(2k^2 + 2k) + 1\)
  - \(\iff a^2 \text{ is odd (since } 2k^2 + 2k \in \mathbb{N}, \text{ by closure)}\)

Since \(a\) was selected arbitrarily, the proposition is true for all \(n \in \mathbb{N}\).
Proofs of Equivalence (if and only if)

• Claim: \((\forall n \in \mathbb{N})[n \text{ is odd } \iff n^2 \text{ is odd}]\)

• Claim: \((\forall n,m \in \mathbb{N})[n \text{ and } m \text{ have the same “parity” } \iff n + m \text{ is even}]\)