Announcements

• Homework #4 is due Tomorrow

• Midterm #1 is on Thursday (Tuesday, 3/10) and will cover material up through (and including) Fundamental Theorem of Arithmetic
Quotient-Remainder Theorem

\[(\forall a \in \mathbb{Z})(\forall n \in \mathbb{Z}^+)(\exists q, r \in \mathbb{Z})[(a = nq + r) \land (0 \leq r < d)]\]

Examples.
Quotient-Remainder Theorem and Proof by Cases

Common proof technique:

**Claim:** \((\forall x \in \mathbb{Z}) \text{ ["Something about mod 7"]}\).

**Proof:**

Let \(a \in \mathbb{Z}\), selected arbitrarily. By the QRT, there exists \(k \in \mathbb{Z}\) satisfying one of these cases:

- case \(a = 7k\):
- case \(a = 7k + 1\):
- case \(a = 7k + 2\):
- case \(a = 7k + 3\):
- case \(a = 7k + 4\):
- case \(a = 7k + 5\):
- case \(a = 7k + 6\):

These cases are exhaustive...
Using Quotient Remainder Theorem

• Claim: For all $n$, $2n^2 + 3n + 2$ is not divisible by 5
• Claim: $(\forall n \in \mathbb{Z}) [3 \nmid n \rightarrow n^2 \equiv_3 1]$
Now we can prove...

Claim: For all $a, b \in \mathbb{N}$, the following are equivalent:

1. $a \equiv_n b$
2. $n \mid (a - b)$
3. $(\exists k \in \mathbb{Z}) [a = b + kn]$

How do we prove several things are equivalent?
Floor and ceiling

• Definitions:
  - For all \( x \in \mathbb{R} \), \( n \in \mathbb{Z} \)
    \[
    \lfloor x \rfloor = n \iff n \leq x < n+1
    \]
  - For all \( x \in \mathbb{R} \), \( n \in \mathbb{Z} \)
    \[
    \lceil x \rceil = n \iff n-1 < x \leq n
    \]
Proofs with Floor/Ceiling

Claim: $(\forall x \in \mathbb{R})(\forall y \in \mathbb{Z})[\lfloor x+y \rfloor = \lfloor x \rfloor + y]$ 

Claim: The floor of $(n/2)$ is either  
   a) $n/2$ when $n$ is even  
   b) $(n-1)/2$ when $n$ is odd
Unit 6

Review of Sequences, Summations and Products
Practice Finding an explicit formula

- Figure out the formula for this sequence:

\[1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots\]
Summation & product notation

- Sum of items specified

\[ \sum_{k=1}^{6} 2^k = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 \]

- Product of items specified

\[ \prod_{k=1}^{5} 2k = 2(1) * 2(2) * 2(3) * 2(4) * 2(5) \]
Variable ending point

- n as the index of the final term

\[ \sum_{k=0}^{n} \frac{k + 1}{n + k} \]

- for n = 2
- for n = 3
Nesting of sum/product notation

• Variations (same or different??):

\[
\sum_{j=1}^{n} \sum_{i=1}^{m_j} Y_{ij}^2 \quad \sum_{j=1}^{n} \left( \sum_{i=1}^{m_j} Y_{ij} \right)^2 \quad \left( \sum_{j=1}^{n} \sum_{i=1}^{m_j} Y_{ij} \right)^2
\]
Telescoping series

\[ \sum_{k=1}^{n} \left( \frac{k}{k+1} - \frac{k+1}{k+2} \right) \]

\[ \prod_{i=1}^{n} \left( \frac{i}{i+1} \right) \]
## Properties

- **Merging and splitting**

\[
\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)
\]

\[
\prod_{k=m}^{n} a_k \ast \prod_{k=m}^{n} b_k = \prod_{k=m}^{n} (a_k \ast b_k)
\]

\[
\sum_{k=m}^{n} a_k = \sum_{k=m}^{i} a_k + \sum_{k=i+1}^{n} a_k
\]

\[
\prod_{k=m}^{n} a_k = \prod_{k=m}^{i} a_k \ast \prod_{k=i+1}^{n} a_k
\]
Properties, con't.

- Distribution

\[ c \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (c \cdot a_k) \]
Factorial

- \( n! = n \times (n - 1) \times (n - 2) \times \ldots \times 2 \times 1 \)

- Definition:
  \[
  0! = 1 \\
  n! = n \times (n - 1)!
  \]