

Practice Problems – Type Systems

Here is the simply typed lambda calculus, extended with integers, and its type system:

$$\begin{aligned} e &::= v \mid x \mid e e \\ v &::= n \mid \lambda x:t.e \\ t &::= int \mid t \rightarrow t \\ A &::= \cdot \mid x:t, A \end{aligned}$$

$$\begin{array}{c} \text{INT} \\ \hline A \vdash n : int \\ \text{VAR} \\ \frac{x \in dom(A)}{A \vdash x : A(x)} \\ \text{LAM} \\ \frac{x:t, A \vdash e : t'}{A \vdash \lambda x:t.e : t \rightarrow t'} \\ \text{APP} \\ \frac{A \vdash e_1 : t \rightarrow t' \quad A \vdash e_2 : t}{A \vdash e_1 e_2 : t'} \end{array}$$

1. Draw derivations showing that the following typing judgments hold:

- (a) $\cdot \vdash 42 : int$
- (b) $y : int \vdash \lambda x: int. y : int \rightarrow int$
- (c) $\cdot \vdash \lambda x: int. \lambda y: int. x : int \rightarrow int \rightarrow int$
- (d) $+ : int \rightarrow int \rightarrow int \vdash (\lambda f: int \rightarrow int. f \ 42) (\lambda x: int. + \ x \ 3) : int$

To save writing effort, you can write i instead of int .

2. Give a simply typed lambda calculus term that is type-incorrect, and yet it does not get stuck at run time. Your term must not be typable by the trivial operation of changing type annotations on parameters. For example, $(\lambda x: int \rightarrow int. x) \ 3$ is not a valid answer.

3. Finally, consider type inference for the simply-typed lambda calculus:

$$\begin{aligned} e &::= v \mid x \mid e e \\ v &::= n \mid \lambda x.e \\ t &::= \alpha \mid int \mid t \rightarrow t \\ A &::= \cdot \mid x:t, A \end{aligned}$$

$$\begin{array}{c} \text{INT} \\ \hline A \vdash n : int \\ \text{VAR} \\ \frac{x \in dom(A)}{A \vdash x : A(x)} \\ \text{LAM} \\ \frac{x:\alpha, A \vdash e : t' \quad \alpha \text{ fresh}}{A \vdash \lambda x.e : \alpha \rightarrow t'} \\ \text{APP} \\ \frac{A \vdash e_1 : t \quad A \vdash e_2 : t' \quad t = t' \rightarrow \beta \quad \beta \text{ fresh}}{A \vdash e_1 e_2 : \beta} \end{array}$$

Draw derivations showing type inference applied to the following terms in the empty type environment; write down a solution to the associated constraints; and write down the fully resolved type of the term. We've done the first one as an example.

(a) $\lambda x.x$

$$\frac{\frac{x:\alpha \vdash x:\alpha}{\cdot \vdash \lambda x.x : \alpha \rightarrow \alpha} \quad \cdot \vdash 42 : int \quad (\alpha \rightarrow \alpha) = (int \rightarrow \beta)}{\cdot \vdash (\lambda x.x) \ 42 : \beta}$$

Solution: $\alpha = \beta = int$.

Type: int .

- (b) $\lambda x. \lambda y. x$
- (c) $\lambda x. \lambda y. x \ y$
- (d) $(\lambda x. \lambda y. x) \ 3 \ 42$