1. (15 pts) Prove that in any undirected graph of \( n \) vertices \( (n \geq 2) \), there are always at least two vertices that have the same degree.

2. (20 pts) Suppose that an undirected graph \( G = (V, E) \) contains two nodes \( s \) and \( t \) such that the distance between \( s \) and \( t \) is strictly greater than \( n/2 \). Show that there must exist some node \( v \), not equal to either \( s \) or \( t \), such that deleting \( v \) from \( G \) destroys all \( s-t \) paths. Give an \( O(m+n) \) algorithm to find such a node \( v \).

3. (20 pts) A cut vertex of an undirected connected graph is one whose removal (and removal of its incident edges) disconnects the graph. Give an \( O(m+n) \) algorithm to find all cut vertices of an undirected connected graph. (Hint: derive a relationship between cut vertices and low values.)

4. (25 pts) Given an undirected graph \( G = (V, E) \), an orientation of \( G \) is an assignment of direction to each edge. (In general, a graph has many orientations, each yielding a directed graph.) We say that a legitimate orientation \( O \) of \( G \) is one with the property that if there was an undirected path from \( u \) to \( v \) in \( G \), then there is a directed path from \( u \) to \( v \) and also from \( v \) to \( u \) in the oriented graph.

   (a) Characterize the class of undirected graphs having legitimate orientations. Prove that your characterization is correct.

   (b) Provide an algorithm that, given an undirected graph, finds a legitimate orientation of it, or if none exists, determines so.

5. (Ungraded.) Give an example of a weighted connected undirected graph \( G = (V, E) \) and a vertex \( v \in V \) such that the MST of \( G \) is necessarily very different from a shortest-path tree rooted at \( v \). Can the two trees be completely disjoint? (Recall that a shortest-path tree is the union over all \( u \) of a shortest path from \( v \) to \( u \).)