1. Construct the following graph: $G^* = (V^*, E^*)$. In this graph, the vertices in $V^*$ correspond to the elephants. Each elephant is mapped to one vertex. We put a directed edge from $e_i$ to $e_j$ iff $w_i < w_j$ and $s_i < s_j$. In other words, this means that $i$ and $j$ can both be in $S$ (the desired set) and $e_i$ weighs less than $e_j$. Node $i$ in $G^*$ has a value of $v_i$.

I hope it is clear that any directed path in this acyclic graph corresponds to a subset satisfying the required property. (Note that every subset satisfying the property forms a valid path in the graph.)

Next note that the path of longest value can be found using dynamic programming. For all nodes of out-degree zero we can compute their optimal values easily (just the value of the node).

Each node $x$ can consider its out-edges $(x, y_i)$ and $\text{OPT}(x) = v_x + \max \text{OPT}(y_i)$. Note that we can compute a node’s OPT value once all the nodes it has edges to have had their OPT values computed. (Hint: perform this computation in decreasing weight order.)

2. See solution in KT.

3. All 3-paths from $s$ to $t$ must go through one of $s$’s neighbors (which is in $X$) and then through one of $t$’s neighbors (which is in $Y$). So look at the bipartite subgraph induced by $(N(s), N(t))$. Every 3-path from $s$ to $t$ must go through an edge in this graph. Thus, the number of 3-paths from $s$ to $t$ is equivalent to the number of edges in this graph. To construct the subgraph takes $O(n + m)$ and to determine the number of edges in this subgraph takes no longer than that. Thus the algorithm runs in $O(n + m)$.

If the problem had asked for all vertex-disjoint $s - t$ paths, then the number of such paths is equivalent to the size of the maximum matching on the subgraph induced by $(N(s), N(t))$. This matching can be found in $O(n'm') = O(nm)$, where $n' = |N(s)| + |N(t)|$, and $m'$ is the number edges between $N(s)$ and $N(t)$.

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1 A subgraph of $G = (V,E)$ induced by a subset $S \subseteq V$ of vertices is the subgraph consisting of nodes in $S$ and edges with both endpoints in $S$. 