Solutions to Midterm Exam 1

Solution 1:
(a) Collider: Could be attached to a wall, preventing objects from passing through it. Trigger: Could be attached to a doorway, signaling whenever some object enters the room.
(b) Any convex combination would work, for example \((1/3)p + (1/3)q + (1/3)r\) (which would yield the centroid).
(c) (iii) Decreasing the persistence parameter will cause the bumps to dampen out more quickly.
(d) (i) AABBs will yield a poor approximation if the object is oriented diagonally, (iii) A sphere is never a good approximation to a thin object.

Solution 2:
(a) The point \(q\) lies above the plane if and only if from \(q\)’s perspective, the points \(p_1, p_2,\) and \(p_3\) appear to be oriented clockwise. Thus, \(q\) lies above the plane if and only if \(\text{orient}(p_1, p_2, p_3, q) < 0\). In terms of the underlying determinant, this can also be expressed as
\[
\begin{vmatrix}
1 & 1 & 1 & 1 \\
p_{1,x} & p_{2,x} & p_{3,x} & q_x \\
p_{1,y} & p_{2,y} & p_{3,y} & q_y \\
p_{1,z} & p_{2,z} & p_{3,z} & q_z
\end{vmatrix} < 0.
\]
(b) We begin by computing a few mutually orthogonal unit vectors. Let \(\vec{v}\) be a unit vector that is pointing to the pilot’s front. Let \(\vec{r}\) be a unit vector that is directed to the pilot’s right, and let \(\vec{u}\) be a vector that is directed up relative to the pilot. Clearly,
\[
\vec{v} = \frac{p_1 - p_4}{\|p_1 - p_4\|} \quad \text{and} \quad \vec{r} = \frac{p_2 - p_3}{\|p_2 - p_3\|}.
\]
The vector \(\vec{u}\) is orthogonal to both of these. We have \(\vec{u} = \vec{r} \times \vec{v}\). (While we could normalize \(\vec{u}\), we do not need to do. This is because \(\vec{v}\) and \(\vec{r}\) are already orthogonal to each other and of unit length, which implies that their cross product will be of unit length.)

The point \(c\) lies \(\ell\) units behind \(p_4\) (along the vector \(-\vec{v}\)) and \(h\) units above (along the vector \(\vec{u}\)). Thus, we have
\[
c = p_4 - \ell\vec{v} + h\vec{u}.
\]
Solution 3:

(a) For \( i \in \{1, 2\} \), \( q_i = (x_i, 0, z_i) \). The minimum distance between \( \ell_1 \) and \( \ell_2 \) is achieved by a line segment that is perpendicular to both segments, that is, a horizontal segment. Thus, the minimum distance between the lines is equal to the distance between these points:

\[
\text{dist}(\ell_1, \ell_2) = ||q_2 - q_1|| = \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2}.
\]

Let’s call this quantity \( \delta_0 \). (See the figure below (a).)

(b) There are three cases in determining the distance between the two segments, depending on the vertical overlap of the two segments.

**Case 1:** If \( y_1^+ < y_2^- \) then the \( y \)-extents of segment 1 are strictly below those of segment 2, and the minimum distance is achieved by the upper point of \( s_1 \) and the lower point of \( s_2 \). Thus \( \delta = ||p_2^- - p_1^+|| = \sqrt{(x_2 - x_1)^2 + (y_2^- - y_1^+)^2 + (z_2 - z_1)^2} \).

**Case 2:** If \( y_1^- > y_2^+ \) then the \( y \)-extents of segment 1 are strictly above those of segment 2, and the minimum distance is achieved by the lower point of \( s_1 \) and the upper point of \( s_2 \). Thus \( \delta = ||p_2^+ - p_1^-|| = \sqrt{(x_2 - x_1)^2 + (y_2^+ - y_1^-)^2 + (z_2 - z_1)^2} \).

**Case 3:** Otherwise the \( y \)-extents overlap \( [y_1^-, y_1^+] \cap [y_2^-, y_2^+] \neq \emptyset \), and the interiors of the two segments share a common \( y \)-value. The distance between the segments is just the distance \( \delta_0 \) between the vertical lines. (See the figure (b).)

(c) Given the distance \( \delta \) between the two segments, it follows that the capsules overlap if and only if \( \delta \leq r_1 + r_2 \).

Solution 4:

(a) Observe that a point represented relative to Earth has its \( x \)-coordinate increased by 10 when expressed relative to the Sun. Thus

\[
T_{[\text{e} \leftarrow \text{s}]} = \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
(b) The inverse simply decreases the $x$-coordinate by 10.

\[
T_{[e\leftarrow s]} = \begin{pmatrix}
1 & 0 & -10 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

(c) Given $v_{[e]}$ we apply the rotation $\text{Rot}(\phi)$ transformation, then convert to Sun coordinate by applying $T_{[s\leftarrow e]}$, and finally apply the revolution $\text{Rot}(\theta)$. Since the point is already in Sun coordinates, no further change is needed. Thus we have

\[
v'_{[s]} = (\text{Rot}(\theta) \cdot T_{[s\leftarrow e]} \cdot \text{Rot}(\phi))v_{[e]}.
\]

The desired transformation is the product of three matrices given in the parentheses.

(d) If $v_{[s]}$ is given instead, we just need to replace $v_{[e]}$ in the above equation with $T_{[e\leftarrow s]}v_{[s]}$. Thus,

\[
v'_{[s]} = (\text{Rot}(\theta) \cdot T_{[s\leftarrow e]} \cdot \text{Rot}(\phi) \cdot T_{[e\leftarrow s]})v_{[s]}.
\]