CMSC 430
Introduction to Compilers
Spring 2016

Type Systems

What is a Type System?

- A *type system* is some mechanism for distinguishing good programs from bad
  - Good programs = well typed
  - Bad programs = ill-typed or not typable

- Examples:
  - 0 + 1 // well typed
  - false 0 // ill-typed: can't apply a boolean
  - 1 + (if true then 0 else false) // ill-typed: can't add boolean to integer
    - Notice that the type system may be *conservative* — it may report programs as erroneous if they could run without type errors

A Definition of Type Systems

"A type system is a tractable syntactic method for proving the absence of certain program behaviors by classifying phrases according to the kinds of values they compute."

– Benjamin Pierce, *Types and Programming Languages*

The Plan

- Start with lambda calculus (yay!)
- Add types to it
  - Simply-typed lambda calculus
- Prove type soundness
  - So we know what our types mean
  - We'll learn about structural induction here
- Discuss issues of types in real languages
  - E.g., null, array bounds checks, etc
- Explain type inference
- Add subtyping (for OO) to all of the above

Lambda calculus

- We'll use lambda calculus are a "core language" to explain type systems
  - Has essential features (functions)
  - No overlapping constructs
  - And none of the cruft
    - Extra features of full language can be defined in terms of the core language ("syntactic sugar")
- We will add features to lambda calculus as we go on

Simply-Typed Lambda Calculus

- e ::= n | x | λx:t.e | e e
  - Functions include the type of their argument
  - We've added integers, so we can have (obvious) type errs
  - We don't really need this, but it will come in handy

- t ::= int | t → t
  - t1 → t2 is a the type of a function that, given an argument of type t1, returns a result of type t2
    - t1 is the *domain*, and t2 is the *range*

Type Judgments

- Our type system will prove *judgments* of the form
  - $A \vdash e : t$
  - "In type environment A, expression e has type t"

Type Environments

- A *type environment* is a map from variables to types (a kind of symbol table)
  - $\cdot$ is the empty type environment
    - A closed term e is *well-typed* if $\cdot \vdash e : t$ for some t
    - We'll abbreviate this as $\vdash e : t$
  - $x{:}t, A$ is just like A, except x now has type t
    - The type of x in $x{:}t, A$ is t
    - The type of $z \neq x$ in $x{:}t, A$ in the type of z in A
- When we see a variable in a program, we look in the type environment to find its type

Type Rules

$$\frac{}{A \vdash n : \text{int}}$$

$$\frac{x \in \text{dom}(A)}{A \vdash x : A(x)}$$

$$\frac{x{:}t, A \vdash e : t'}{A \vdash \lambda x{:}t.e : t \to t'}$$

$$\frac{A \vdash e1 : t \to t' \quad A \vdash e2 : t}{A \vdash e1\ e2 : t'}$$

Example

$A = - : \text{int} \rightarrow \text{int}$

$$\dfrac{\dfrac{- \in \text{dom}(A)}{A \vdash - : \text{int} \rightarrow \text{int}} \qquad A \vdash 3 : \text{int}}{A \vdash -\ 3 : \text{int}}$$

Another Example

$A = + : \mathrm{int} \to \mathrm{int} \to \mathrm{int}$
$B = x : \mathrm{int}, A$

$$\dfrac{\dfrac{\dfrac{\dfrac{+ \in \mathrm{dom}(B)}{B \vdash + : i \to i \to i} \qquad \dfrac{x \in \mathrm{dom}(B)}{B \vdash x : i}}{B \vdash +\ x : \mathrm{int} \to \mathrm{int}} \qquad B \vdash 3 : \mathrm{int}}{\dfrac{B \vdash +\ x\ 3 : \mathrm{int}}{A \vdash (\lambda x{:}\mathrm{int}.+\ x\ 3) : \mathrm{int} \to \mathrm{int}}} \qquad A \vdash 4 : \mathrm{int}}{A \vdash (\lambda x{:}\mathrm{int}.+\ x\ 3)\ 4 : \mathrm{int}}$$

We'd usually use infix $x + 3$

An Algorithm for Type Checking

- Our type rules are deterministic
  - For each syntactic form, only one possible rule

- They define a natural type checking algorithm
  - TypeCheck : type env × expression → type

```
TypeCheck(A, n) = int
TypeCheck(A, x) = if x in dom(A) then A(x) else fail
TypeCheck(A, λx:t.e) = TypeCheck((A, x:t), e)
TypeCheck(A, e1 e2) =
    let t1 = TypeCheck(A, e1) in
    let t2 = TypeCheck(A, e2) in
       if dom(t1) = t2 then range(t1) else fail
```

Semantics

- Here is a small-step, call-by-value semantics
  - If an expression can't be evaluated any more and is not a value, then it is *stuck*

$$\frac{}{(\lambda x.e1)\ v2 \rightarrow e1[v2\backslash x]} \qquad \frac{e1 \rightarrow e1'}{e1\ e2 \rightarrow e1'\ e2}$$

$$\frac{e2 \rightarrow e2'}{v1\ e2 \rightarrow v1\ e2'}$$

$e ::= v \mid x \mid e\ e$

$v ::= n \mid \lambda x{:}t.e$ *values* – not evaluated

Progress

- Suppose $\cdot \vdash e : t$. Then either e is a value, or there exists e' such that $e \to e'$
- Proof by induction on e
  - Base cases n, λx.e – these are values, so we're done
  - Base case x – can't happen (empty type environment)
  - Inductive case e1 e2 – If e1 is not a value, then by induction we can evaluate it, so we're done, and similarly for e2. Otherwise both e1 and e2 are values. Inspection of the type rules shows that e1 must have a function type, and therefore must be a lambda since it's a value. Therefore we can make progress.

Preservation

- If $\cdot \vdash e : t$ and $e \rightarrow e'$ then $\cdot \vdash e' : t$
- Proof by induction on $e \rightarrow e'$
  - Induction (easier than the base case!). Expression e must have the form e1 e2.
  - Assume $\cdot \vdash e1\ e2 : t$ and $e1\ e2 \rightarrow e'$. Then we have $\cdot \vdash e1 : t' \rightarrow t$ and $\cdot \vdash e2 : t'$.
  - Then there are three cases.
    - If $e1 \rightarrow e1'$, then by induction $\cdot \vdash e1 : t' \rightarrow t$, so $e1'\ e2$ has type t
    - If reduction inside e2, similar

Preservation, cont'd

- Otherwise $(\lambda x.e)\ v \to e[v\backslash x]$. Then we have

$$\frac{x{:}\,t' \vdash e : t}{\vdash \lambda x.e : t' \to t}$$

  - Thus we have
    - $x : t' \vdash e : t$
    - $\cdot \vdash v : t'$
  - Then by the substitution lemma (not shown) we have
    - $\cdot \vdash e[v\backslash x] : t$
  - And so we have preservation

Substitution Lemma

- If $A \vdash v : t$ and $x{:}t, A \vdash e : t'$, then $A \vdash e[v\backslash x] : t'$
- Proof: Induction on the structure of e
- For lazy semantics, we'd prove
  - If $A \vdash e1 : t$ and $x{:}t, A \vdash e : t'$, then $A \vdash e[e1\backslash x] : t'$

Soundness

- So we have
  - Progress: Suppose $\cdot \vdash e : t$. Then either e is a value, or there exists e' such that $e \rightarrow e'$
  - Preservation: If $\cdot \vdash e : t$ and $e \rightarrow e'$ then $\cdot \vdash e' : t$
- Putting these together, we get soundness
  - If $\cdot \vdash e : t$ then either there exists a value v such that $e \rightarrow^* v$, or e diverges (doesn't terminate).
- What does this mean?
  - Evaluation getting stuck is bad, so
  - "Well-typed programs don't go wrong"

Consequences of Soundness

- Progress—anything that can go wrong "locally" at run time should be forbidden in the type system
  - E.g., can't "call" an int as if it were a function
  - To check this, identify all places where the semantics get stuck, and cross-reference with type rules
- Preservation—running a program can't change types
  - E.g., after beta reduction, types still the same
  - To check this, ensure that for each possible way the semantics can take a step, types are preserved
- These problems greatly influence the way type systems are designed

Conditionals

e ::= ... | true | false | if e then e else e

$$\frac{}{A \vdash \text{true} : \text{bool}} \qquad \frac{}{A \vdash \text{false} : \text{bool}}$$

$$\frac{A \vdash \text{e1} : \text{bool} \quad A \vdash \text{e2} : \text{t} \quad A \vdash \text{e3} : \text{t}}{A \vdash \text{if e1 then e2 else e3} : \text{t}}$$

Conditionals (op sem)

e ::= ... | true | false | if e then e else e

$$\frac{}{\text{if true then e2 else e3} \rightarrow \text{e2}}$$

$$\frac{}{\text{if false then e2 else e3} \rightarrow \text{e3}}$$

$$\frac{\text{e1} \rightarrow \text{e1'}}{\begin{array}{c}\text{if e1 then e2 else e3} \rightarrow \\ \text{if e1' then e2 else e3}\end{array}}$$

- Notice how need to satisfy progress and preservation influences type system, and interplay between operational semantics and types

Product Types (Tuples)

e ::= ... | (e, e) | fst e | snd e

$$\frac{A \vdash e1 : t \quad A \vdash e2 : t'}{A \vdash (e1,e2) : t \times t'}$$

$$\frac{A \vdash e : t \times t'}{A \vdash \text{fst } e : t} \qquad \frac{A \vdash e : t \times t'}{A \vdash \text{snd } e : t'}$$

- Or, maybe, just add functions
  - pair : $t \to t' \to t \times t'$
  - fst : $t \times t' \to t$
  - snd : $t \times t' \to t'$

Sum Types (Tagged Unions)

$e ::= \ldots \mid \mathrm{inL}_{t2}\ e \mid \mathrm{inR}_{t1}\ e$

$\quad \mid (\mathrm{case}\ e\ \mathrm{of}\ x1{:}t1 \to e1 \mid x2{:}t2 \to e2)$

$$\frac{A \vdash e : t1}{A \vdash \mathrm{inL}_{t2}\ e : t1 + t2} \qquad \frac{A \vdash e : t2}{A \vdash \mathrm{inR}_{t1}\ e : t1 + t2}$$

$$\frac{\begin{array}{c} A \vdash e : t1 + t2 \\ x1{:}t1, A \vdash e1 : t \quad x2{:}t2, A \vdash e2 : t \end{array}}{A \vdash (\mathrm{case}\ e\ \mathrm{of}\ x1{:}t1 \to e1 \mid x2{:}t2 \to e2) : t}$$

Self Application and Types

- Self application is not checkable in our system

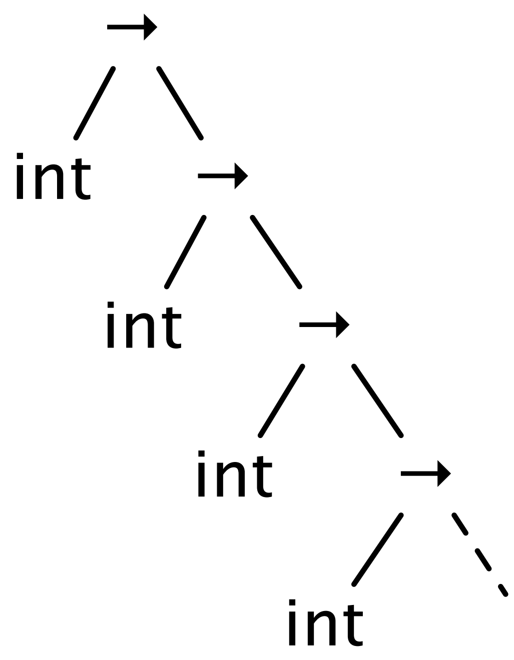
$$\dfrac{\dfrac{x{:}?,A \vdash x : t \to t' \qquad x{:}?,A \vdash x : t}{x{:}?,A \vdash x\ x : \ldots}}{A \vdash \lambda x{:}?.x\ x : \ldots}$$

  - It would require a type t such that $t = t \to t'$
    - (We'll see this next, but so far...)

- The simply-typed lambda calculus is *strongly normalizing*
  - Every program has a normal form
  - I.e., every program halts!

Recursive Types

- We can type self application if we have a type to represent the solution to equations like $t = t \rightarrow t'$
  - We define the type $\mu\alpha.t$ to be the solution to the (recursive) equation $\alpha = t$
  - Example: $\mu\alpha.\mathrm{int} \rightarrow \alpha$



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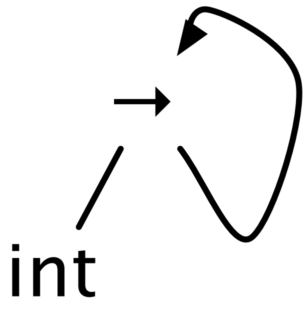


Discussion

- In the pure lambda calculus, every term is typable with recursive types
  - (Pure = variables, functions, applications only)

- Most languages have some kind of “recursive” type
  - E.g., for data structures like lists, tree, etc.

- However, usually two recursive types that define the same structure but use a different name are considered different
  - E.g., in C, struct foo { int x; struct foo *next; } is different from struct bar { int x; struct bar *next; }

Subtyping

- The Liskov Substitution Principle (paraphrased):

  Let q(x) be a property provable about objects x of type T. If S is a subtype of T, then q(y) should be provable for objects y of type S.

- In other words

  If S is a subtype of T, then an S can be used anywhere a T is expected

- Common used in object-oriented programming
  - Subclasses can be used where superclasses expected
  - This is a kind of *polymorphism*

Kinds of Polymorphism

- Parametric polymorphism
  - Generics in Java, `a types in OCaml
- Another popular form is subtype polymorphism
  - As in OO programming
  - These two can be combined (c.f. Java)
- Some languages also have *ad-hoc polymorphism*
  - E.g., + operator that works on ints and floats
  - E.g., overloading in Java

Lambda Calc with Subtyping

- e ::= n | f | x | λx:t.e | e e
  - We now have both floating point numbers and integers
  - We want to be able to implicitly use an integer wherever a floating point number is expected
  - Warning: This is a bad design! Don't do this in real life
- t ::= int | float | t → t
  - We want int to be a subtype of float

Subtyping

- We'll write $t1 \le t2$ if $t1$ is a subtype of $t2$
- Define subtyping by more inference rules
- Base case

$$\frac{}{\text{int} \le \text{float}}$$

  - (notice reverse is not allowed)
- What about function types?

$$\frac{???}{t1 \to t1' \le t2 \to t2'}$$

Replacing "f x" by "g x"

- Suppose g : $t1 \to t1'$ and f : $t2 \to t2'$
- When is $t1 \to t1' \leq t2 \to t2'$?

- Return type:
  - We are expecting $t2'$ (f's return type)
  - So we can return *at most* $t2'$
  - So need $t1' \leq t2'$
- Examples
  - If we're expecting float, can return int or float
  - If we're expecting int, can only return int

Replacing "f x" by "g x"

- Suppose $g : t1 \rightarrow t1'$ and $f : t2 \rightarrow t2'$
- When is $t1 \rightarrow t1' \leq t2 \rightarrow t2'$?

- Argument type:
  - We are supposed to accept expecting t2 (f's arg type)
  - So we must accept *at least* t2
  - So need $t2 \leq t1$
- Examples
  - A function that accepts an int can be replaced by one that accepts int, or one that accepts float
  - A function that accepts a float can only be replaced by one that accepts float

Subtyping on Function Types

$$\frac{t2 \le t1 \quad t1' \le t2'}{t1 \to t1' \le t2 \to t2'}$$

- We say that arrow is
  - *Covariant* in the range (subtyping dir the same)
  - *Contravariant* in the domain (subtyping dir flips)
- Some languages have gotten this wrong
  - Eiffel allows covariant parameter types

Similar Pattern for Pre/Post-conds

- class A { int f(int x) { ... } }
- class B extends A { int f(int x) { ... } }

- A.f — precondition Pre_A, postcondition Post_A
- B.f — precondition Pre_B, postcondition Post_B
- Relationship among {Pre,Post}_{A,B}?
  - Post_A ⇒ Post_B
  - Pre_B ⇒ Pre_A
- Example:
  - Pre_A = (x > 42), Post_A = (ret > 42)
  - Pre_B = (x > 0), Post_B = (ret > 100)

Type Rules, with Subtyping

$$\frac{}{A \vdash n : \mathrm{int}} \qquad \frac{}{A \vdash f : \mathrm{float}}$$

$$\frac{x \in \mathrm{dom}(A)}{A \vdash x : A(x)} \qquad \frac{x{:}t, A \vdash e : t'}{A \vdash \lambda x{:}t.e : t \rightarrow t'}$$

$$\frac{A \vdash e1 : t1 \rightarrow t1' \qquad A \vdash e2 : t2 \qquad t2 \leq t1}{A \vdash e1\ e2 : t1'}$$

Soundness

- Progress and preservation still hold
  - Slight tweak: as evaluation proceeds, expression's type may "decrease" in the subtyping sense
  - Example:
    - (if true then n else f) : float
    - But after taking one step, will have type int $\leq$ float

- Proof: exercise for the reader

Subtyping, again

$$\frac{}{A \vdash n : \text{int}} \qquad \frac{}{A \vdash f : \text{float}}$$

$$\frac{x \in \text{dom}(A)}{A \vdash x : A(x)} \qquad \frac{x{:}t, A \vdash e : t'}{A \vdash \lambda x{:}t.e : t \rightarrow t'}$$

$$\frac{A \vdash e1 : t1 \rightarrow t1' \qquad A \vdash e2 : t2}{A \vdash e1\ e2 : t1'}$$

$$\frac{A \vdash e : t \qquad t \leq t'}{A \vdash e : t'}$$

Subtyping, again (cont'd)

- Rule with subtyping is called *subsumption*
  - Very clearly captures subtyping property
- But system is no longer *syntax driven*
  - Given an expression e, there are two rules that apply to e ("regular" type rule, and subsumption rule)
- Can prove that the two systems are equivalent
  - Exercise left to the reader

Lambda Calc with Updatable Refs

- e ::= ... | ref e | !e | e := e
  - ML-style updatable references
    - ref e — allocate memory and set its contents to e; return pointer
    - !e — dereference pointer and return contents
    - e1 := e2 — update contents pointed to by e1 with e2
- t ::= ... | t ref
  - A t ref is a pointer to contents of type t

Type Rules for Refs

$$\frac{A \vdash e : t}{A \vdash \text{ref } e : t \text{ ref}} \qquad \frac{A \vdash e : t \text{ ref}}{A \vdash !e : t}$$

$$\frac{A \vdash e1 : t1 \text{ ref} \quad A \vdash e2 : t2 \quad t2 \leq t1}{A \vdash e1 := e2 : t1}$$

Subtyping Refs

- The wrong rule for subtyping refs is

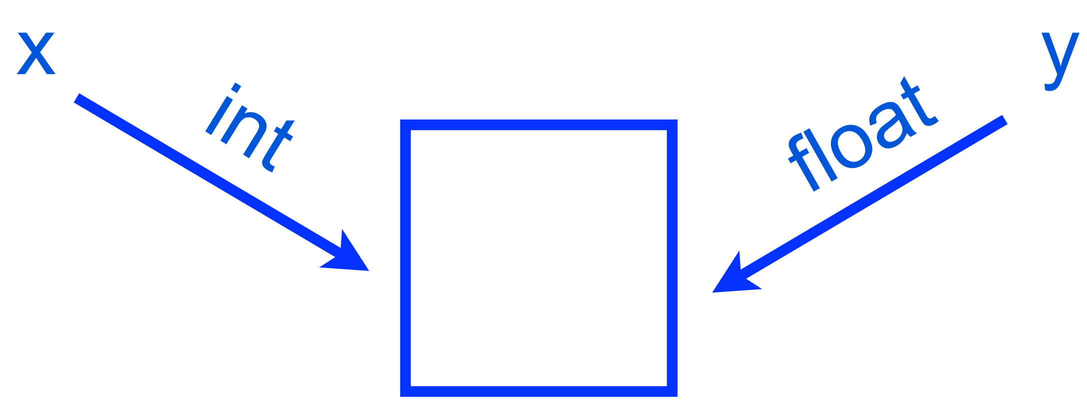
$$\frac{t1 \leq t2}{t1 \text{ ref} \leq t2 \text{ ref}}$$

- Counterexample

```
let x = ref 3 in   (* x : int ref *)
let y = x in       (* y : float ref *)
y := 3.14          (* oops! !x is now a float *)
```

Aliasing

- We have multiple names for the same memory location
  - But they have different types
  - This we can **write** into the same memory at different types



Solution #1: Java's Approach

- Java uses this subtyping rule
  - If S is a subclass of T, then S[] is a subclass of T[]

- Counterexample:
  - Foo[] a = new Foo[5];
  - Object[] b = a;
  - b[0] = new Object();   // forbidden at runtime
  - a[0].foo();   // …so this can't happen

Solution #2: Purely Static

- Reason from rules for functions
  - A reference is like an object with two methods:
    - get : unit $\rightarrow$ t
    - set : t $\rightarrow$ unit
  - Notice that t occurs both co- and contravariantly
  - Thus it is *non-variant*
- The right rule:

$$\frac{t1 \leq t2 \qquad t2 \leq t1}{t1 \text{ ref} \leq t2 \text{ ref}} \quad \text{or} \quad \frac{t1 = t2}{t1 \text{ ref} \leq t2 \text{ ref}}$$

Type Inference

- Let's consider the simply typed lambda calculus with integers
  - e ::= n | x | λx:t.e | e e

- *Type inference*: Given a bare term (with no type annotations), can we reconstruct a valid typing for it, or show that it has no valid typing?

Type Language

- Problem: Consider the rule for functions

$$\frac{x{:}t, A \vdash e : t'}{A \vdash \lambda x{:}t.e : t \to t'}$$

- Without type annotations, where do we get t?
  - We'll use *type variables* to stand for as-yet-unknown types
    - $t ::= \alpha \mid \text{int} \mid t \to t$
  - We'll generate *equality constraints* $t = t$ among the types and type variables
    - And then we'll solve the constraints to compute a typing

Type Inference Rules

$$\frac{}{A \vdash n : int}$$

$$\frac{x \in dom(A)}{A \vdash x : A(x)}$$

$$\frac{x{:}\alpha, A \vdash e : t' \qquad \alpha \text{ fresh}}{A \vdash \lambda x.e : \alpha \to t'}$$

$$\frac{A \vdash e1 : t1 \qquad A \vdash e2 : t2 \qquad \boxed{t1 = t2 \to \beta} \qquad \beta \text{ fresh}}{A \vdash e1\ e2 : \beta}$$

“Generated” constraint

Example

$$\dfrac{\dfrac{x{:}\alpha,\ A \vdash x{:}\alpha}{A \vdash (\lambda x.x) : \ \alpha \rightarrow \alpha} \qquad A \vdash 3 : \mathrm{int} \qquad \alpha \rightarrow \alpha = \mathrm{int} \rightarrow \beta}{A \vdash (\lambda x.x)\ 3 : \beta}$$

- We collect all constraints appearing in the derivation into some set C to be solved
- Here, C contains just $\alpha \rightarrow \alpha = \mathrm{int} \rightarrow \beta$
  - Solution: $\alpha = \mathrm{int} = \beta$
- Thus this program is typable, and we can derive a typing by replacing α and β by int in the proof tree

Solving Equality Constraints

- We can solve the equality constraints using the following rewrite rules, which reduce a larger set of constraints to a smaller set
  - $C \cup \{int=int\} \Rightarrow C$
  - $C \cup \{\alpha=t\} \Rightarrow C[t\backslash\alpha]$
  - $C \cup \{t=\alpha\} \Rightarrow C[t\backslash\alpha]$
  - $C \cup \{t1 \rightarrow t2=t1' \rightarrow t2'\} \Rightarrow C \cup \{t1=t1'\} \cup \{t2=t2'\}$
  - $C \cup \{int=t1 \rightarrow t2\} \Rightarrow$ unsatisfiable
  - $C \cup \{t1 \rightarrow t2=int\} \Rightarrow$ unsatisfiable

Termination

- We can prove that the constraint solving algorithm terminates.
- For each rewriting rule, either
  - We reduce the size of the constraint set
  - We reduce the number of “arrow” constructors in the constraint set
- As a result, the constraint always gets “smaller” and eventually becomes empty
  - A similar argument is made for strong normalization in the simply-typed lambda calculus

Occurs Check

- We don't have recursive types, so we shouldn't infer them
- So in the operation $C[t\backslash\alpha]$, require that $\alpha \notin FV(t)$
  - (Except if t = a, in which case there's no recursion in the types, so unification should succeed)
- In practice, it may better to allow $\alpha \in FV(t)$ and do the occurs check at the end
  - But that can be awkward to implement

Unifying a Variable and a Type

- Computing C[t\α] by substitution is inefficient

- Instead, use a union-find data structure to represent equal types
  - The terms are in a union-find forest
  - When a variable and a term are equated, we union them so they have the same ECR (equivalence class representative)
    - Want the ECR to be the concrete type with which variables have been unified, if one exists. Can read off solution by reading the ECR of each set.

Example

$\alpha = \text{int} \rightarrow \beta$

$\gamma = \text{int} \rightarrow \text{int}$

$\alpha = \gamma$

Unification

- The process of finding a solution to a set of equality constraints is called *unification*
  - Original algorithm due to Robinson
    - But his algorithm was inefficient
  - Often written out in different form
    - See Algorithm W
  - Constraints usually solved on-line
    - As type inference rules applied

Discussion

- The algorithm we've given finds the *most general type* of a term
  - Any other valid type is "more specific," e.g.,
    - $\lambda x.x : \text{int} \rightarrow \text{int}$
  - Formally, any other valid type can be gotten from the most general type by applying a substitution to the type variables

- This is still a *monomorphic* type system
  - α stands for "some particular type, but it doesn't matter exactly which type it is"

Benefits of Type Inference

- Handles higher-order functions
- Handles data structures smoothly
- Works in infinite domains
  - Set of types is unlimited
- No forward/backward distinction
  - (Compare to data flow analysis, next)

Drawbacks to Type Inference

- Flow-insensitive
  - Types are the same at all program points
  - May produce coarse results
  - Type inference failure can be hard to understand

- Polymorphism may not scale
  - Exponential in worst case
  - Seems fine in practice (witness ML)