Stable Marriage and Man-Optimality

Definition 1. A man $m$ and a woman $w$ are valid partners means there exists a stable matching in which they are paired with each other.

Definition 2. For every man $m$, $m$’s best valid partner (denoted $\text{best}(m)$) is the highest-ranked valid partner of $m$, with respect to $m$’s preference list.

Definition 3. A matching $S$ is man-optimal means that each man $m$ is paired with $\text{best}(m)$ in $S$.

We have already seen that the Gale-Shapley algorithm (GS) always returns a matching that is stable. We will need that to prove the following.

Claim 1. For any fixed rule dictating the order of proposals, the matching $S^*$ returned by GS is man-optimal.

Proof. First note that since $S^*$ is stable, everyone is paired with one of their valid partners. Suppose by way of contradiction that there exists a man who is not paired with his best valid partner. Then he must be paired with a valid partner who comes after his best valid partner in his preference list. In other words, if $S^*$ is not man-optimal, then at least one man was rejected by his best valid partner during the execution of GS.

Consider the first time it happens that some man $m$ is rejected by his best valid partner, $w = \text{best}(m)$: $w$ rejects $m$ to be (or to continue to be) with someone else $m'$ whom she prefers to $m$. Let us call this episode Event X.

Since $w$ and $m$ are valid partners, there exists a stable matching $S'$ in which $w$ is paired with $m$. In $S'$, $m'$ is paired with someone else, say $w' \neq w$. $w'$ is a valid partner of $m'$ since $S'$ is stable.

Now consider what the execution of GS tells us about $m'$’s preference between $w$ and $w'$. Event X was the first time in the execution of GS that any man was rejected by his best valid partner. In particular, at the time that Event X occurred, both the following are true:

- $m'$ has not been rejected by his $\text{best}(m')$ and therefore has not been rejected by any of his valid partners, in particular $w'$;
- $m'$ is paired with $w$, i.e. $m'$ was rejected by every woman before $w$ in his preference list.

Therefore, the execution of GS tells us that $w'$ must be after $w$ in $m'$’s preference list, i.e. $m'$ prefers $w$ to $w'$.

However, this contradicts the stability of $S'$: $(m, w), (m', w') \in S'$, but both $w$ and $m'$ prefer each other to their respective partners in $S'$. Therefore, our initial assumption that some man is rejected by his best valid partner during the execution of GS is false, i.e. $S^*$ is man-optimal.

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