CMSC 858C, Spring 2017, University of Maryland, HW 1
Due as a PDF file by email to Khoa Trinh by 11:59PM on Feb 14th, 2017

Instructions. Please work on this with your group and submit a neatly typed (or handwritten) file to Khoa: one submission per group. Use of other resources including books and the Web is not allowed. In case you can make partial progress on a problem, please write your approach clearly instead of giving no answer.

This HW. The "first-moment method" refers to only using the expectation of a random variable; Markov's inequality is a good example. Similarly, the second-moment method uses the mean and the variance, as typified by Chebyshev's inequality. We will explore these two methods further here.

1. (i) Use the first-moment method to prove the following: there is a constant $c > 0$ such that for any undirected graph G – with number of edges denoted m as usual – the probability that the simple "random cut" algorithm produces a cut of size at least $0.499m$, is at least c. (ii) How will you use this result if you want an algorithm to construct a cut of size at least $0.499m$ with probability at least 0.9?

2. Now use the second-moment method to show that the probability in 1(i) is actually as high as $1-O(1/m)$. The point of this problem, as compared to problem 1, is that we can get (significantly) better bounds as we go to higher moments, i.e., as we use more information about the underlying random variable.

3. (i) For any random variable X with some mean μ and variance v, and for any $a > 0$, prove that each of $\Pr[X \leq \mu - a]$ and $\Pr[X \geq \mu + a]$ is upper-bounded by $v/(v + a^2)$. Second, in the context of Chebyshev's inequality (where we want to upper-bound $\Pr[|X - \mu| \geq a]$), is it a good idea to combine these two to get an alternative bound? Why, or why not? (ii) A median for a random variable X is any value u such that $\Pr[X \leq u] \geq 1/2$ and $\Pr[X \geq u] \geq 1/2$. Show that for any random variable X with some mean μ and standard deviation σ, its median u is such that $\mu - \sigma \leq u \leq \mu + \sigma$.

4. Let $p_n(k)$ be the number of permutations of the set $\{1, 2, \ldots, n\}$, which have exactly k fixed points. Prove that $\sum_{k=0}^{n} k \cdot p_n(k) = n!$. (An element i is called a fixed point of the permutation f if $f(i) = i$.)

5. With the same notation as problem 4, determine the value of $\sum_{k=0}^{n} k^2 p_n(k)$.

6. We are given a box with a red balls and b blue balls, with $n = a + b$. We pick n balls at random without replacement, one-by-one; let X_i be the indicator random variable for the i'th ball drawn being red. (i) Show that for all i, $\Pr[X_i = 1] = a/(a + b)$. (ii) Prove that the X_i's are "pairwise negatively correlated": if $i \neq j$, then $\mathbf{E}[X_i X_j] \leq (a/(a + b))^2$ by explicitly computing this expectation. (**Hint:** Symmetry, and forgetting about the colors for a moment, can help with both questions.)