CMSC 858C, Spring 2017, University of Maryland, HW 2

Due as a PDF file by email to Khoa Trinh by 11:59PM on Mar 2nd, 2017

Instructions. Please work on this with your group and submit a neatly typed (or handwritten) file to Khoa: one submission per group. Use of other resources including books and the Web is not allowed. In case you can make partial progress on a problem, please write your approach clearly instead of giving no answer.

This HW. We primarily study tail bounds and sampling here.

0. **Required Reading.** Read *http://www.cs.cmu.edu/afs/cs/academic/class/15859-f04/www/scribes/lec8.pdf*. It gives random sampling as a powerful tool to even approximate $\#P$-complete problems; in Section 8.3, it gives a "plus 1 / minus 1" trick very similar to Alon-Matias-Szegedy – and then discusses the limitations of this approach in the context there (approximating non-negative permanents) due to the high variance.

1. Prove that truncated inclusion-exclusion alternately lower- and upper- bounds $\Pr[E_1 \vee E_2 \vee \cdots \vee E_n]$ where the E_i are arbitrary given events. (**Hint:** Use one of the two ways we discussed of viewing a joint probability distribution. The identity $\sum_{i=0}^{r}(-1)^i\binom{n}{i} = (-1)^r\binom{n-1}{r}$ (for integers $0 \le r \le n$, and with $\binom{n-1}{n}$ taken to be zero) could help.)

2. Consider a Poisson random variable X with mean λ. Use the exponential moment-generating function approach discussed for Chernoff (i.e., bounding $\mathbf{E}[e^{tX}]$ for an optimally-chosen t) to upper-bound $\Pr[X \ge \lambda(1+\delta)]$ with $\delta > 0$. Do you get a familiar bound?

The next three problems study the following setup, where we have some random variables $X_1, X_2, \ldots, X_n$ whose sum X we want tail-bounds for, but where the X_i's are not fully independent. We will assume that each X_i lies in $[0,1]$ and that $\mathbf{E}[X] = \mu$. Importantly, the X_i's have a dependency graph G – an undirected graph on the vertices $\{1, 2, \ldots, n\}$ – where informally, an edge between i and j indicates that X_i and X_j are dependent, and the absence of such an edge means that X_i and X_j are independent. Formally, if S is any independent set in G, then the random variables $(X_i : i \in S)$ are mutually independent of each other. Let Δ denote the maximum degree of G.

3. Use the second-moment method to show that for any $a > 0$, $\Pr[|X - \mu| \ge a] \le O(\mu\Delta/a^2)$.

4. Why stop at the second moment? In this problem and the next, we try for Chernoff-Hoeffding-type bounds, under the assumption that for some p, $\mathbf{E}[X_i] = p$ for all i. We will use the following useful result on graph coloring (lookup "graph coloring" if you need to): *the vertices of the graph G can be properly colored using $\Delta+1$ colors in such a way that each color is used in total by either $\lfloor n/(\Delta+1)\rfloor$ or $\lceil n/(\Delta+1)\rceil$ vertices.*

Let $CH^+(\alpha, \beta)$ be the following Chernoff-Hoeffding upper-tail bound: suppose, for any m, any independent $Y_1, Y_2, \ldots, Y_m$ with $Y_i \in [0,1]$, and $Y = \sum_i Y_i$ with $\mathbf{E}[Y] \ge \alpha$, we have $\Pr[Y \ge \alpha(1+\beta)] \le CH^+(\alpha, \beta)$ for $\beta > 0$.

Show that for any $\delta > 0$, $\Pr[X \ge \mu(1+\delta)] \le (\Delta+1) \cdot CH^+(\lfloor n/(\Delta+1)\rfloor \cdot p, \delta)$.

5. Again under the assumption that $\mathbf{E}[X_i] = p$ for all i, formulate and prove an analog of problem 4 for $\Pr[X \le \mu(1-\delta)]$, with, of course, $\delta > 0$ again.