CMSC 858C, Spring 2017, University of Maryland, HW 4
Due as a PDF file by email to Khoa Trinh by 9PM on May 8th

Instructions. Please work on this with your group and submit a neatly typed (or handwritten) file to Khoa: one submission per group. Use of other resources including books and the Web is not allowed. In case you can make partial progress on a problem, please write your approach clearly instead of giving no answer.

1. Consider a random graph G sampled from the $G(n,p)$ model for some $p = p(n)$. Let A be the event that G contains a Hamiltonian cycle, and B be the event that G is colorable using at most 100 colors. Is it necessarily true that $\Pr[A \wedge B] \leq \Pr[A] \cdot \Pr[B]$?

2. Show that there is some constant $C > 0$ such that the following holds. Suppose $L \geq 3$ is some given constant, and that we choose a random graph G from the $G(n,p)$ model, where $p \geq C(n \ln n)^{1/L}/n$. Then (if the constant C has been chosen appropriately large), show that $\Pr[G \text{ has diameter at most } L] \to 1$ as $n \to \infty$, using the following approach. Consider any fixed pair of distinct vertices u, v, and let

$$q = \Pr[G \text{ has no path of length at most } L \text{ connecting } u \text{ and } v].$$

Show, using Janson's inequality, that $n^2 \cdot q \to 0$ as $n \to \infty$. (Note that once this is shown, the union bound will do the job for us. Also, there are other ways of proving this diameter bound. However, for the purpose of further familiarizing you with Janson's inequality, you are asked to give a proof along the above lines.)

3. Show that there is a constant $a > 0$ such that the following holds. We have an arbitrary graph $G = (V, E)$ with maximum degree Δ. Each vertex v also has a list of colors L_v; we want to color each vertex v with some color from L_v, so that we get a proper coloring (i.e., no two adjacent vertices get the same color). Prove that this is possible if the following holds: there is a non-negative value $b_{v,c}$ for all vertices v and all colors $c \in L_v$, such that:

- $\forall v,\ \sum_{c \in L_v} b_{v,c} = 1$; and
- $\forall (u,v) \in E,\ \sum_{c \in L_u \cap L_v} b_{u,c} \cdot b_{v,c} \leq a/\Delta$.

Try to get as large a value for the constant a as you can.

4. Suppose we have a Δ-regular graph $G = (V, E)$; you can assume that Δ is "large", i.e., that for any one constant Δ_0 that you choose, you can assume $\Delta \geq \Delta_0$. We also have the set of colors $S = \{1, 2, \ldots, \Delta + 1\}$, and a given permutation σ of V. Consider the following random process where we try to color several vertices legally using the set of colors S (in addition to maintaining some other properties seen below): each vertex v chooses a *tentative* color T_v uniformly at random from S and independently, and if *no neighbor u of v that precedes v in σ has $T_u = T_v$*, then T_v becomes the permanent color of v – otherwise, v does not get colored.

a. Let X_v be the indicator for the event that v gets colored. Show that $\mathbf{E}[X_v] \geq 1/e$.

b. Let Y_v be the number of neighbors of v that get colored successfully; i.e. $Y_v = \sum_{u:(u,v) \in E} X_u$. Prove that for any given $\epsilon > 0$,

$$\Pr[Y_v \leq (1-\epsilon)\Delta/e] \leq \exp(-\Omega(\Delta\epsilon^2)).$$

(Use a Martingale tail bound. Be careful with the calculations.)

c. Let $A_v(t)$ be the bad event that "there exists some color c that is received by at least $t \log \Delta / \log\log \Delta$ neighbors of v at the end of the above random process". Prove that for any constant $a > 0$, there is a constant $b > 0$ such that $\Pr[A_v(b)] \leq \Delta^{-a}$.

d. Let $B_v(t, \epsilon)$ be the bad event that "either $A_v(t)$ occurs, or $Y_v \leq (1-\epsilon)\Delta/e$". Use parts (b) and (c) along with a technique we have learnt, to show that there exist constants $b > 0$ and $\epsilon \in (0, 1)$ (i.e., these two parameters are independent of Δ) such that with positive probability, our random process avoids all of the bad events $\{B_v(b, \epsilon) : \ v \in V\}$.

e. Convert the "positive probability" in part (d) to a randomized polynomial-time algorithm that succeeds with high probability (just give brief high-level details).

f. Suppose the permutation σ had been chosen at random. What would the constant "$1/e$" in part (a) get improved to?