Theorem (Block & Novikoff, 1962)

If the training data \( D = \{(x_1, y_1), \ldots, (x_N, y_N)\} \) is **linearly separable** with margin \( \gamma \) by a unit norm hyperplane \( w_* (\|w_*\| = 1) \) with \( b = 0 \), then **perceptron training converges** after \( \frac{R^2}{\gamma^2} \) errors during training (assuming \( \|x\| < R \) for all \( x \)).

**Proof:**

- Margin of \( w_* \) on any *arbitrary example* \((x_n, y_n)\): \[ \frac{y_n w_*^T x_n}{\|w_*\|} = y_n w_*^T x_n \geq \gamma \]
- Consider the \((k + 1)^{th}\) mistake: \( y_n w_k^T x_n \leq 0 \), and update \( w_{k+1} = w_k + y_n x_n \)
- \( w_{k+1}^T w_* = w_k^T w_* + y_n w_*^T x_n \geq w_k^T w_* + \gamma \) (why is this nice?)
- Repeating iteratively \( k \) times, we get \( w_{k+1}^T w_* > k\gamma \) \hspace{1cm} (1)
- \( \|w_{k+1}\|^2 = \|w_k\|^2 + 2y_n w_k^T x_n + \|x\|^2 \leq \|w_k\|^2 + R^2 \) (since \( y_n w_k^T x_n \leq 0 \))
- Repeating iteratively \( k \) times, we get \( \|w_{k+1}\|^2 \leq kR^2 \) \hspace{1cm} (2)
Theorem (Block & Novikoff, 1962)

If the training data $D = \{(x_1, y_1), \ldots, (x_N, y_N)\}$ is linearly separable with margin $\gamma$ by a unit norm hyperplane $w_\ast (||w_\ast|| = 1)$ with $b = 0$, then perceptron training converges after $\frac{R^2}{\gamma^2}$ errors during training (assuming $(||x|| < R)$ for all $x$).

What does this mean?

- Perceptron converges quickly when margin is large, slowly when it is small
- Bound does not depend on number of training examples $N$, nor on number of features $d$
- Proof guarantees that perceptron converges, but not necessarily to the max margin separator
What you should know

- Perceptron concepts
  - training/prediction algorithms (standard, voting, averaged)
  - convergence theorem and what practical guarantees it gives us
  - how to draw/describe the decision boundary of a perceptron classifier
- Fundamental ML concepts
  - Determine whether a data set is linearly separable and define its margin
  - Error driven algorithms, online vs. batch algorithms
Expressivity

- Many functions are linear
  - Conjunctions:
    - \( y = x_1 \cap x_3 \cap x_5 \)
    - \( y = \text{sign}(1 \times x_1 + 1 \times x_3 + 1 \times x_5 - 3), \ w = [1,0,1,0,1] \)
  - At least \( m \) of \( n \):
    - \( y = \text{at least 2 of } (x_1, x_3, x_5) \)
- Many functions are not
  - Xor: \( y = x_1 \cap x_2 \cup \neg x_1 \cap \neg x_2 \)
  - Non trival DNF: \( y = x_1 \cap x_2 \cup x_3 \cap x_4 \)
- But can be made linear
Functions Can be Made Linear

- Data are not linearly separable in one dimension
- Not separable if you insist on using a specific class of functions
Blown Up Feature Space

Data are separable in $<x, x^2>$ space

- Key issue: Representation:
  - what features to use.
  - Computationally, can be done implicitly (kernels)
  - Not always ideal.
Exclusive-OR (XOR)

- \( y = x_1 \land x_2 \lor \neg x_1 \land \neg x_2 \)

- In general: a parity function.

- \( x_i \in \{0,1\} \)

- \( f(x_1, x_2, \ldots, x_n) = 1 \)
  iff \( \sum x_i \) is even

This function is not linearly separable.
Practical Issues

• “garbage in, garbage out”
  • Learning algorithms can’t compensate for useless training examples
    • E.g., if all features are irrelevant
  • Feature design can have bigger impact on performance than tweaking the learning algorithm
THIS IS YOUR MACHINE LEARNING SYSTEM?

YUP! YOU POUR THE DATA INTO THIS BIG PILE OF LINEAR ALGEBRA, THEN COLLECT THE ANSWERS ON THE OTHER SIDE.

WHAT IF THE ANSWERS ARE WRONG?

JUST STIR THE PILE UNTIL THEY START LOOKING RIGHT.
## Practical Issues

<table>
<thead>
<tr>
<th>Classifier</th>
<th>Accuracy on test set</th>
</tr>
</thead>
<tbody>
<tr>
<td>Team A</td>
<td>80.00</td>
</tr>
<tr>
<td>Team B</td>
<td>79.90</td>
</tr>
<tr>
<td>Team C</td>
<td>79.00</td>
</tr>
<tr>
<td>Team D</td>
<td>78.00</td>
</tr>
</tbody>
</table>

Which classifier is the best?

- This result table alone cannot give us the answer
- Solution: statistical hypothesis testing
## Practical Issues

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Is the difference in accuracy between A and B statistically significant?

What is the probability that the observed difference in performance was due to chance?
A confidence of 95%

• Does NOT mean
  “There is a 95% chance than classifier A is better than classifier B”

• It means
  “If I run this experiment 100 times, I expect A to perform better than B 95 times.”
Practical Issues: Debugging!

- You’ve implemented a learning algorithm
- You try it on some train/dev/test data
- But it doesn’t seem to learn

What’s going on?
- Is the data too noisy?
- Is the learning problem too hard?
- Is the implementation of the learning algorithm buggy?
Strategies for Isolating Causes of Errors

- Is the problem with **generalization** to test data?
  - Can learner fit the training data?
  - Yes: problem is in generalization to test data
  - No: problem is in representation (need better features or better data)

- **Train/test mismatch?**
  - Try reselecting train/test by shuffling training data and test together
• Strategies for Isolating Causes of Errors

• Is algorithm **implementation correct**?
  • Measure loss rather than accuracy
  • Hand-craft a toy dataset

• **Is representation adequate?**
  • Can you learn if you add a cheating feature that perfectly correlates with correct class?

• **Do you have enough data?**
  • Try training on 80% of the training set, how much does it hurt performance?
Algorithm 8 **CrossValidate**(*LearningAlgorithm, Data, K*)

1. $\hat{e} \leftarrow \infty$ \hspace{1cm} // store lowest error encountered so far  
2. $\hat{\alpha} \leftarrow \text{unknown}$ \hspace{1cm} // store the hyperparameter setting that yielded it  
3. for all hyperparameter settings $\alpha$ do  
   4. $\text{err} \leftarrow [\ ]$ \hspace{1cm} // keep track of the $K$-many error estimates  
      5. for $k = 1$ to $K$ do  
         6. $\text{train} \leftarrow \{(x_n, y_n) \in \text{Data} : n \mod K \neq k - 1\}$  
         7. $\text{test} \leftarrow \{(x_n, y_n) \in \text{Data} : n \mod K = k - 1\}$ \hspace{1cm} // test every $K$th example  
         8. $\text{model} \leftarrow \text{Run LearningAlgorithm on train}$  
         9. $\text{err} \leftarrow \text{err} \oplus \text{error of model on test}$ \hspace{1cm} // add current error to list of errors  
      end for  
   end for  
10. $\text{avgErr} \leftarrow \text{mean of set err}$  
11. if $\text{avgErr} < \hat{e}$ then  
    12. $\hat{e} \leftarrow \text{avgErr}$ \hspace{1cm} // remember these settings  
    13. $\hat{\alpha} \leftarrow \alpha$ \hspace{1cm} // because they’re the best so far  
14. end if  
15. end for
N-fold cross validation

Instead of a single test-training split:

- Split data into $N$ equal-sized parts

- Train and test $N$ different classifiers

- Report average accuracy and standard deviation of the accuracy
Improving Input Representations

- Feature pruning
- Feature normalization

Centering:

\[ x_{n,d} \leftarrow x_{n,d} - \mu_d \]  \hspace{1cm} (5.1)

Variance Scaling:

\[ x_{n,d} \leftarrow x_{n,d} / \sigma_d \]  \hspace{1cm} (5.2)

Absolute Scaling:

\[ x_{n,d} \leftarrow x_{n,d} / r_d \]  \hspace{1cm} (5.3)

where:

\[ \mu_d = \frac{1}{N} \sum_n x_{n,d} \]  \hspace{1cm} (5.4)

\[ \sigma_d = \sqrt{\frac{1}{N-1} \sum_n (x_{n,d} - \mu_d)^2} \]  \hspace{1cm} (5.5)

\[ r_d = \max_n |x_{n,d}| \]  \hspace{1cm} (5.6)

- Example normalization

\[ x_n \leftarrow x_n / \|x_n\| \]
Practical Issues: Debugging!

• You probably have a bug
  • if the learning algorithm cannot overfit the training data
  • if the predictions are incorrect on a toy 2D dataset hand-crafted to be learnable
Practical Issues: Evaluation, beyond accuracy

- So far we’ve measured classification performance using **accuracy**

- But this is not a good metric when some errors matter more than others
  - Given medical record, predict whether patient has cancer or not
  - Given a document collection and a query, find documents in collection that are relevant to query
The 2-by-2 contingency table

Imagine we are addressing a document retrieval task for a given query, where +1 means that the document is relevant and -1 means that the document is not relevant.

We can categorize predictions as:
- true/false positives
- true/false negatives

<table>
<thead>
<tr>
<th>Prediction</th>
<th>Gold label = +1</th>
<th>Gold label = -1</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>tp</td>
<td>fp</td>
</tr>
<tr>
<td>-1</td>
<td>fn</td>
<td>tn</td>
</tr>
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</table>
**Precision**: % of positive predictions that are correct

**Recall**: % of positive gold labels that are found

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A combined measure: F

- A combined measure that assesses the P/R tradeoff is F measure

\[
F = \alpha \frac{1}{P} + (1 - \alpha) \frac{1}{R} = \frac{(\beta^2 + 1)PR}{\beta^2P + R}
\]

- People usually use balanced F-1 measure
  - i.e., with \( \beta = 1 \) (that is, \( \alpha = \frac{1}{2} \))
  - Harmonic mean \( F = \frac{2PR}{P+R} \)
Formalizing Errors

The learned classifier

$\mathcal{F}$ set of all possible classifiers using a fixed representation

$\text{error}(f) = \left[ \text{error}(f) - \min_{f^* \in \mathcal{F}} \text{error}(f^*) \right] + \left[ \min_{f^* \in \mathcal{F}} \text{error}(f^*) \right]$}

estimation error

approximation error

How far is the learned classifier $f$ from the optimal classifier $f^*$?

Quality of the model family aka hypothesis class
The bias/variance trade-off

- **Trade-off between**
  - approximation error (bias)
  - estimation error (variance)

- **Example:**
  - Consider the always positive classifier
    - Low variance as a function of a random draw of the training set
    - Strongly biased toward predicting +1 no matter what the input
Recap: practical issues

- Learning algorithm is only one of many steps in designing a ML application
- Many things can go wrong, but there are practical strategies for
  - Improving inputs
  - Evaluating
  - Tuning
  - Debugging
- Fundamental ML concepts: estimation vs. approximation error
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