1. Convolution of two Gaussian produces another Gaussian which is described by the following parameters: $\mu = \mu_1 + \mu_2$ and $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$. That means mean is shifted and the distribution becomes more spread.

2. Yes they can. Consider $u = (2, 3)$, $v = (-1, 0)$, $w = (2, -6)$. In that case,
   \begin{align*}
u.w &= (−1, 0). (2,−6) = −2 + 0 = −2 < 0 \\
u.w &= (2, 3). (2,−6) = 4 − 18 = −14 < 0
\end{align*}

3. **Proof.** In that case, sum of the row vectors of $A − cI$ is 0 since it subtracts $c$ from an element in each column. If we add a row to another one, the determinant does not change. So if we add first $n−1$ rows to the $n^{th}$ row, all elements in the $n^{th}$ row become 0. Therefore the determinant becomes 0 and $c$ is an eigenvalue of $A$ since it satisfies the characteristic equation $\det(A - \lambda I) = 0$. ■

4. **Proof.** If $\lambda$ is an eigenvalue of $A$ and $X$ is the corresponding eigenvector, we have $AX = \lambda X$. Therefore for any scalar $s$,
\begin{align*}
AX − sX &= \lambda X − sX \\
\Rightarrow (A − sI)X &= (\lambda − s)X.
\end{align*}

By the definition of eigenvalue and eigenvector, we can conclude that $\lambda − s$ is an eigenvalue of $A − sI$ for any scalar $s$ and $X$ is the corresponding eigenvector. ■

5. (a) True. We have $(AB)^T = B^T A^T = BA$. We need to show that $A$ and $A^T$ have the same eigenvalues. We know that
\begin{align*}
(A - \lambda I)^T &= A^T - \lambda I \\
\Rightarrow det(A^T - \lambda I) &= det((A - \lambda I)^T) = det(A - \lambda I)
since det(A) = det(A^T).
\end{align*}

(b) True. Suppose $\lambda$ is an eigenvalue for $AB$ with corresponding eigenvector $v$. So we have $\lambda Bv = B\lambda v = B(ABv) = (BA)Bv$. Thus $\lambda$ is also an eigenvalue for $BA$ with the corresponding eigenvector $Bv$. (Since $B$ is invertible, $Bv$ is not the zero vector unless $v = 0.$)
(c) False. Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. In that case B is invertible. Eigenvectors of $AB$ are $(-0.7578, 0.6525)^T$ and $(-0.4031, -0.9151)^T$ while eigenvectors of $BA$ are $(-0.8289, 0.5593)^T$ and $(-0.5949, -0.8038)^T$.

6. (a) Yes. If the rows of $A$ are linearly independent, then $\det(A) \neq 0$. In that case $A$ is invertible, so $x = A^{-1}b$ is a solution. Another way to think of this is in such a case $\text{rank}(A) = m$, so column vectors of $A$ span $\mathbb{R}^m$. Since $b \in \mathbb{R}^m$, it can be represented by the linear combination of column vectors of $A$.

(b) No. If $m < n$, there will be infinitely many solutions since there will be free variables.

7. Proof. We have $Av = \lambda v \implies v = A^{-1}\lambda v \implies v = \lambda A^{-1}v \implies \frac{1}{\lambda}v = A^{-1}v$. ■