Due: February 8st, 2018
Date: January 28, 2018

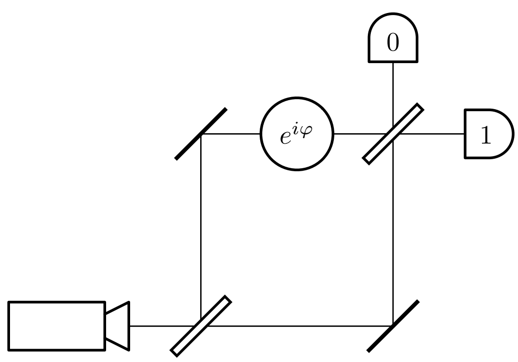
Assignment 1

Please submit it electronically to ELMS. This assignment is 7% in your total points. For the simplicity of the grading, the total points for the assignment is 70.

Problem 1. *Quantum Circuits*

1. *(10 points)* The CCCNOT (triple-controlled NOT) gate is a four-bit reversible gate that flips its fourth bit if and only if the first three bits are all in the state 1. Show how to implement a CCCNOT gate using Toffoli gates. You may use additional workspace as needed. You may assume that bits in the workspace start with a particular value, either 0 or 1, provided you return them to that value.
2. *(Bonus: 10 points)* Show that a Toffoli gate cannot be implemented using any number of CNOT gates, with any amount of workspace. Hence the CNOT gate alone is not universal. (Hint: It may be helpful to think of the gates as performing arithmetic operations on integers mod 2.)

Problem 2. *Mach-Zehnder interferometer with a phase shift.*



Analyze the experiment depicted above using the mathematical model described in class. (Note that the model from class differs slightly from the model described in the textbook; in particular, you should use the matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to model the beamsplitters.)

1. *(7 points)* Compute the quantum state of the system just before reaching the detectors. Express your answer using Dirac notation.
2. *(3 points)* Compute the probability that the "0" detector clicks as a function of φ, and plot your result for $\varphi \in [0, 2\pi]$.

Problem 3.

1. *(4 points)* Express each of the three Pauli operators,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

   using Dirac notation in the computational basis.

2. *(5 points)* Find the eigenvalues and the corresponding eigenvectors of each Pauli operator. Express the eigenvectors using Dirac notation.

3. *(3 points)* Write the operator $X \otimes Z$ as a matrix and using Dirac notation (in both cases using the computational basis).

4. *(3 points)* What are the eigenspaces of the operator $X \otimes Z$? Express them using Dirac notation.

5. *(5 points)* Using the Spectral Decomposition show that $HXH^\dagger = Z$, where

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Problem 4. *Unitary operations and measurements.* Consider the state

$$|\psi\rangle = \frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle.$$

1. *(3 points)* Let $|\phi\rangle = (I \otimes H)|\psi\rangle$, where H denotes the Hadamard gate. Write $|\phi\rangle$ in the computational basis.

2. *(4 points)* Suppose the first qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the second qubit?

3. *(4 points)* Suppose the second qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the first qubit?

4. *(4 points)* Suppose $|\phi\rangle$ is measured in the computational basis. What are the probabilities of the four possible outcomes? Show that they are consistent with the marginal probabilities you computed in the previous two parts.

Problem 5. Let θ be a fixed, known angle. Suppose someone flips a fair coin and, depending on the outcome, either gives you the state

$$|0\rangle \qquad \text{or} \qquad \cos\theta|0\rangle + \sin\theta|1\rangle$$

(but does not tell you which).

1. *(5 points)* Consider measuring the given state in the orthonormal basis consisting of

$$|\phi\rangle = \cos\phi|0\rangle + \sin\phi|1\rangle, \qquad |\phi^\perp\rangle = \sin\phi|0\rangle - \cos\phi|1\rangle$$

   Find the probabilities of all the possible measurement outcomes for each possible value of the given state

2. *(5 points)* Calculate the probability of correctly distinguisihing the two possible states using the above measurement (In terms of ϕ)

3. *(5 points)* Calculate the optimal value of ϕ in order to best distinguish the states. What is the optimal success probability?
