CMSC/PHYS 457, Introduction to Quantum Computing
Spring 2018, University of Maryland

Due: February 22nd, 2018
Date: February 7, 2018

Assignment 2

Please submit it electronically to ELMS. This assignment is 7% in your total points. For the simplicity of the grading, the total points for the assignment is 70.

Problem 1. *Entanglement swapping.* Alice and Bob would like to share the entangled state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Unfortunately, they do not initially share any entanglement, but fortunately, they have a mutual friend, Charlie. Alice shares a copy of $|\beta_{00}\rangle$ with Charlie, and Bob also shares a copy of $|\beta_{00}\rangle$ with Charlie.

1. *(Points :2)* Write the initial state using Dirac notation in the computational basis, with the first qubit belonging to Alice, the second and third qubits belonging to Charlie, and the fourth qubit belonging to Bob.

2. *(Points :5)* Suppose Charlie performs a Bell measurement on his two qubits (one of which is entangled with Alice and the other of which is entangled with Bob). For each possible measurement outcome, give the probability with which it occurs and the resulting post-measurement state for Alice and Bob.

3. *(Points :3)* Describe a protocol whereby Charlie sends a classical message to Alice, and Alice processes her quantum state, such that Alice and Bob share the state $|\beta_{00}\rangle$ at the end of the protocol.

Problem 2. *The Hadamard gate and qubit rotations*

1. *(Points :3)* Suppose that $(n_x, n_y, n_z) \in \mathbb{R}^3$ is a unit vector and $\theta \in \mathbb{R}$. Show that

$$e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)} = \cos(\tfrac{\theta}{2})\, I - i\sin(\tfrac{\theta}{2})\,(n_x X + n_y Y + n_z Z).$$

2. *(Points :5)* Find a unit vector $(n_x, n_y, n_z) \in \mathbb{R}^3$ and numbers $\phi, \theta \in \mathbb{R}$ so that

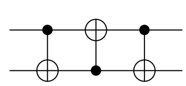
$$H = e^{i\phi} e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)},$$

where H denotes the Hadamard gate. What does this mean in terms of the Bloch sphere?

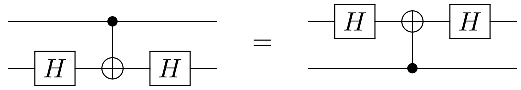
3. *(Points :7)* Write the Hadamard gate as a product of rotations about the x and y axes. In particular, find $\alpha, \beta, \gamma, \phi \in \mathbb{R}$ such that $H = e^{i\phi} R_y(\gamma) R_x(\beta) R_y(\alpha)$.

Problem 3. *Circuit identities.*

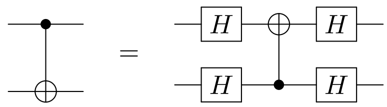
1. *(Points :3)* Show that the following circuit swaps two qubits:



2. *(Points :3)* Verify the following circuit identity:



3. *(Points :4)* Verify the following circuit identity:

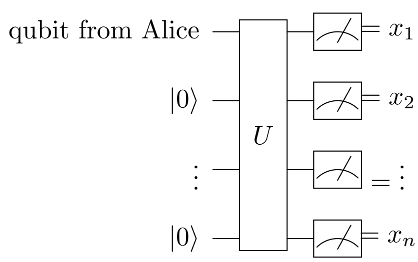


   Give an interpretation of this identity.

Problem 4. *Universality of gate sets.* Prove that each of the following gate sets either is or is not universal. You may use the fact that the set $\{\textsc{cnot}, H, T\}$ is universal.

1. *(Points :5)* $\{H, T\}$
2. *(Points :5)* $\{\textsc{cnot}, T\}$
3. *(Points :5)* $\{\textsc{cnot}, H\}$
4. *(Bonus Points: 10)* $\{\textsc{c}Z, K, T\}$, where $\textsc{c}Z$ denotes a controlled-Z gate and $K = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$

Problem 5. *A qubit cannot be used to communicate a trit perfectly* Suppose that Alice wants to convey a trit of information (an element of $\{0, 1, 2\}$) to Bob and all she is allowed to do is prepare one qubit and send it to Bob. Bob is allowed to prepare $n - 1$ additional qubits, each in state $|0\rangle$, and apply an n-qubit unitary U operation to the entire n qubit system followed by a measurement in the computational basis.



The outcome will be an element of $\{0, 1\}^n$. It is conceivable that such a scheme could exist where Bob can determine the trit from these n bits (e.g., by a function $f(x_1, \cdots, x_n) \in \{0, 1, 2\}$). We shall prove that this is impossible.

The framework is that Alice starts with a trit $j \in \{0, 1, 2\}$ (unknown to Bob) and, based on j, prepares a one-qubit state, $\alpha_j|0\rangle + \beta_j|1\rangle, j \in \{0, 1, 2\}$. and sends it to Bob.

Then Bob applies some n-qubit unitary U to $(\alpha_j|0\rangle + \beta_j|1\rangle)|00\cdots 0\rangle$ and measures each qubit in the computational basis, obtaining some $x \in \{0, 1\}^n$ as outcome. Finally, Bob applies some function $f : \{0, 1\}^n \to \{0, 1, 2\}$ to x to obtain a trit. The scheme *works* if and only if, starting with any $j \in \{0, 1, 2\}$, the resulting x will satisfy $f(x) = j$ with probability 1.

1. *(Points :5)* Note that each row of the matrix U is a 2^n-dimensional vector. For $j \in \{0, 1, 2\}$, define the space V_j to be the span of all rows of U that are indexed by an element of the set $f^{-1}(j) \subseteq \{0, 1\}^n$. Prove that V_0, V_1, and V_2 are mutually orthogonal spaces.

2. *(Points :5)* Explain why, for a scheme to work, $(\alpha_j|0\rangle + \beta_j|1\rangle)|00\cdots0\rangle \in V_j$ must hold for all $j \in \{0,1,2\}$.
3. *(Points :5)* Prove that it is impossible for $(\alpha_j|0\rangle + \beta_j|1\rangle)|00\cdots0\rangle \in V_j$ to hold for all $j \in \{0,1,2\}$.

Problem 6. *Asymptotic notation.* Indicate whether the following statements are true or false.

1. *(Points :1)* $10n^3 + 3n^2 \in O(n^4)$
2. *(Points :1)* $10n^3 + 3n^2 \in \Omega(n^4)$
3. *(Points :1)* $1/n^2 \in O(1/n)$
4. *(Points :1)* $(2^n)^2 \in O(2^n)$
5. *(Points :1)* $(2^n)^2 \in 2^{\Theta(n)}$
