Assignment 5

Please submit it electronically to ELMS. This assignment is 7% in your total points. For the simplicity of the grading, the total points for the assignment is 70.

Problem 1. *Density matrices.* Consider the ensemble in which the state $|0\rangle$ occurs with probability 3/5 and the state $(|0\rangle + |1\rangle)/\sqrt{2}$ occurs with probability 2/5.

1. *(Points :3)* What is the density matrix ρ of this ensemble?
2. *(Points :3)* Write ρ in the form $\frac{1}{2}(I + r_x X + r_y Y + r_z Z)$, and plot ρ as a point in the Bloch sphere.
3. *(Points :3)* Suppose we measure the state in the computational basis. What is the probability of getting the outcome 0? Compute this both by averaging over the ensemble of pure states and by computing $\mathrm{Tr}(\rho|0\rangle\langle 0|)$, and show that the results are consistent.
4. *(Points :4)* How does the density matrix change if we apply the Hadamard gate? Compute this both by applying the Hadamard gate to each pure state in the ensemble and finding the corresponding density matrix, and by computing $H\rho H^\dagger$.

Problem 2. *Local operations and the partial trace.*

1. *(Points :4)* Let $|\psi\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle$. Let ρ denote the density matrix of $|\psi\rangle$ and let ρ' denote the density matrix of $(I \otimes H)|\psi\rangle$. Compute ρ and ρ'.
2. *(Points :3)* Compute $\mathrm{Tr}_B(\rho)$ and $\mathrm{Tr}_B(\rho')$, where B refers to the second qubit.
3. *(Points :4)* Let ρ be a density matrix for a quantum system with a bipartite state space $A \otimes B$. Let I denote the identity operation on system A, and let U be a unitary operation on system B. Prove that $\mathrm{Tr}_B(\rho) = \mathrm{Tr}_B((I \otimes U)\rho(I \otimes U^\dagger))$.
4. *(Bonus Points: 3)* Show that the converse of part (c) holds for pure states. In other words, show that if $|\psi\rangle$ and $|\phi\rangle$ are bipartite pure states, and $\mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \mathrm{Tr}_B(|\phi\rangle\langle\phi|)$, then there is a unitary operation U acting on system B such that $|\phi\rangle = (I \otimes U)|\psi\rangle$.
5. *(Bonus Points: 2)* Does the converse of part (c) hold for general density matrices? Prove or disprove it.

Problem 3. *Product and entangled states.* Determine which of the following states are entangled. If the state is not entangled, show how to write it as a tensor product; if it is entangled, prove this.

1. *(Points :4)* $\frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle$
2. *(Points :5)* $\frac{1}{2}(|00\rangle - i|01\rangle + i|10\rangle + |11\rangle)$
3. *(Points :5)* $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$

Problem 4. *A nonlocal game.* Consider the game where Alice and Bob are physically separated and their goal is to produce outputs that satisfy the winning conditions specified below. Alice and Bob receive $s, t \in \{0, 1, 2\}$ as input (s to Alice and t to Bob), at which point they are forbidden from communicating with each other (so Alice has no idea what t is and Bob has no idea what s is). They each output a bit, a for Alice and b for Bob. The winning conditions are:

- $a = b$ in the case where $s = t$.
- $a \neq b$ in the case where $s \neq t$.

1. *(Points :5)* Show that any classical strategy (that uses no quantum information) of Alice and Bob that always succeeds in the $s = t$ cases can succeed with probability at most 2/3 in the $s \neq t$ cases.
2. *(Points :7)* Give a quantum strategy (that is, one where Alice and Bob can create an entangled state before the game starts and then base their outcomes on their measurements of their parts of this state) that always succeeds in the $s = t$ cases and succeeds with probability 3/4 in the $s \neq t$ cases. (Hint: try the entangled state $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$ and have Alice and Bob perform rotations depending on s and t respectively.)

Problem 5. *The five-qubit code.* Consider a quantum error correcting code that encodes one logical qubit into five physical qubits, with the logical basis states

$$\begin{aligned}
|0_L\rangle = \tfrac{1}{4}(&|00000\rangle \\
&+|10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle + |00101\rangle \\
&-|11000\rangle - |01100\rangle - |00110\rangle - |00011\rangle - |10001\rangle \\
&-|01111\rangle - |10111\rangle - |11011\rangle - |11101\rangle - |11110\rangle) \\
|1_L\rangle = \tfrac{1}{4}(&|11111\rangle \\
&+|01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle + |11010\rangle \\
&-|00111\rangle - |10011\rangle - |11001\rangle - |11100\rangle - |01110\rangle \\
&-|10000\rangle - |01000\rangle - |00100\rangle - |00010\rangle - |00001\rangle).
\end{aligned}$$

1. *(Points :6)* Show that $|0_L\rangle$ and $|1_L\rangle$ are simultaneous eigenstates (with eigenvalue +1) of the operators given in equation 10.5.18 of KLM. (Hint: You can show this without explicitly checking every case.)
2. *(Points :8)* Show that this code can correct an X or Z error acting on any of the five qubits. You should explain how the different possible errors would be reflected by a measurement of the error syndrome.
3. *(Points :3)* Explain why this means that the code can correct any single-qubit error.
4. *(Points :3)* Find logical Pauli operators X_L and Z_L such that $X_L|0_L\rangle = |1_L\rangle$, $X_L|1_L\rangle = |0_L\rangle$, $Z_L|0_L\rangle = |0_L\rangle$, and $Z_L|1_L\rangle = -|1_L\rangle$.
5. *(Bonus Points: 5)* Give a quantum circuit that computes the syndrome of the five-qubit code.
