Here is a list of example problems on the linear algebra that we will use in quantum information. You don’t need to submit solutions for these problems, However, you probably want to figure out answers to these problems as well.

**Problem 1.1.** For complex number \( c = a + bi \), recall that the real and imaginary parts of \( c \) are denoted \( \text{Re}(c) = a \) and \( \text{Imag}(c) = b \).

- Prove that \( c + c^* = 2 \cdot \text{Re}(c) \).
- Prove that \( cc^* = a^2 + b^2 \).
- What is the polar form of \( c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \)? Use the fact that \( e^{i\theta} = \cos \theta + i \sin \theta \).
- Draw \( c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \) as a vector in the complex plane.

**Solution 1.1.**

- Prove that \( c + c^* = 2 \cdot \text{Re}(c) \).
  
  Proof:
  
  \[
  c + c^* = (a + bi) + (a - bi) = 2 \cdot \text{Re}(c)
  \]

- Prove that \( cc^* = a^2 + b^2 \).
  
  Proof:
  
  \[
  cc^* = (a + bi) \times (a - bi) = a^2 + abi - abi - i^2b^2 = a^2 + b^2
  \]

- What is the polar form of \( c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \)? Use the fact that \( e^{i\theta} = \cos \theta + i \sin \theta \).
  
  Proof:
  
  \[
  e^{i\theta} = \cos \theta + i \sin \theta = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i
  \]
  \[
  \cos \theta = \frac{1}{\sqrt{2}}, \sin \theta = \frac{1}{\sqrt{2}}, \theta = \frac{\pi}{4} + 2k\pi, k \in \mathbb{N}
  \]
  \[
  \therefore e^{i\theta} = e^{i\left(\frac{\pi}{4}\right)}
  \]

- Draw \( c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \) as a vector in the complex plane.
Problem 1.2. Define that
\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]

- What is \( \text{tr}(A |1\rangle \langle 0|) \)? (Hint: This can be computed quickly by using the cyclic property of the trace and the outer product representation of \( A \). Do master this trick; it will be used repeatedly in the course and save you much time.)

- Let \( |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \). Use the same trick above, along with the fact that the trace is linear, to quickly evaluate \( \text{tr}(A \cdot |+\rangle \langle +|) \).

Solution 1.2.
- \( b \)
- \( \frac{1}{4} (a + b + c + d) \)

Problem 1.3.
- Write out the 4-dimensional vector for \( \langle \alpha |0\rangle + \beta |1\rangle \otimes (\gamma |0\rangle + \delta |1\rangle) \)?

- Let \( B_1 = \{|\psi_1\rangle, |\psi_2\rangle\}, B_2 = \{|\phi_1\rangle, |\phi_2\rangle\} \) be two orthonormal bases for \( \mathbb{C}^2 \). Can you construct an orthonormal basis for \( \mathbb{C}^4 \)?

Solution 1.3.
- \[ \begin{pmatrix} \alpha \gamma \\ \alpha \delta \\ \beta \gamma \\ \beta \delta \end{pmatrix} \]
\[ B_3 = \{ |\psi_1\rangle \otimes |\phi_1\rangle, |\psi_1\rangle \otimes |\phi_2\rangle, |\psi_2\rangle \otimes |\phi_1\rangle, |\psi_2\rangle \otimes |\phi_2\rangle \} \]

**Problem 1.4.**

- Write out the $4 \times 4$ matrix representing $Y \otimes Y$.
- Prove that $(Z \otimes Y)^\dagger = Z \otimes Y$. Do not write out any matrices explicitly; rather, you must use the properties of the tensor product, dagger, and $Y$.

**Solution 1.4.**

\[
\begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\]

\[ \therefore (Z \otimes Y)^\dagger = Z^\dagger \otimes Y^\dagger \text{ and } Z^\dagger = Z, Y^\dagger = Y \therefore (Z \otimes Y)^\dagger = Z \otimes Y \]

**Problem 2.1.**

- Write down a matrix that is not Hermitian.
- Let $A \in L(C^d)$ be a Hermitian matrix. Prove that if for all $|\psi\rangle \in C^d$, $\langle \psi | A | \psi \rangle \geq 0$, then $A$ has only non-negative eigenvalues.
- Let $A \in L(C^d)$ be a Hermitian matrix. Prove that if $A$ has only non-negative eigenvalues, then for all $|\psi\rangle \in C^d$, $\langle \psi | A | \psi \rangle \geq 0$.

**Solution 2.1.**

\[
\begin{pmatrix}
1 & i \\
1 - i & 1
\end{pmatrix}
\]

- Let $|\psi\rangle$ be the eigenvector of $A$

\[ \therefore A = A^\dagger, \langle \psi | A | \psi \rangle \geq 0 \]

\[ \therefore \langle \psi | A | \psi \rangle = \langle \psi | (A | \psi \rangle = \lambda \langle \psi | \psi \rangle = \lambda \geq 0 \]

(1)

- Given $A$ is Hermitian, let $|\psi_i\rangle$ be $A$’s eigenvector with eigenvalue $\lambda_i \geq 0$ for $i \in [d]$. Thus, for any $|\psi\rangle \in C^d$, we have

\[ |\psi\rangle = \sum_i \alpha_i |\psi_i\rangle . \]

Then we have

\[ A |\psi\rangle = \sum_i \alpha_i A |\psi_i\rangle = \sum_i \alpha_i \lambda_i |\psi_i\rangle . \]

Thus,

\[ \langle \psi | A | \psi \rangle = \sum_i |\alpha_i|^2 \lambda_i \geq 0. \]
Problem 2.2. Given $|\rangle$ state, and suppose that we measure in the computational basis $B = \{|0\rangle\langle0|, |1\rangle\langle1|\}$. What are the probabilities for each possible measurement outcome, and the corresponding post-measurement states?

Solution 2.2. The probability of outcome 0 with post-measurement state $|0\rangle$ is 0.5, and the probability of outcome 1 with post-measurement state $|1\rangle$ is also 0.5.

Problem 2.3.

- Let $A, B \in L(C^d)$ be positive semi-definite matrices. Prove that $A + B$ is positive semi-definite.
- Prove that if $\rho$ and $\sigma$ are density matrices, then so is $p_1\rho + p_2\sigma$ for any $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$.

Solution 2.3.

- For any $|\psi\rangle$, we have

\[
\langle \psi | A | \psi\rangle \geq 0, \langle \psi | B | \psi\rangle \geq 0 \\
\therefore \langle \psi | (A + B) | \psi\rangle = \langle \psi | A | \psi\rangle + \langle \psi | B | \psi\rangle \geq 0 \tag{2}
\]

- It suffices to show that $p_1\rho + p_2\sigma$ is positive semi-definite (which is basically implied by the first item) and has trace 1. The later follows from

\[
\text{tr}(p_1\rho + p_2\sigma) = p_1 \text{tr}(\rho) + p_2 \text{tr}(\sigma) = p_1 + p_2 = 1.
\]

Problem 2.4. Let $|\psi\rangle = \alpha_0|a_0\rangle|b_0\rangle + \alpha_1|a_1\rangle|b_1\rangle$ be the Schmidt decomposition of a two-qubit state $|\psi\rangle$. Prove that for any single qubit unitaries $U$ and $V$, $|\psi\rangle$ is entangled if and only if $|\psi'\rangle = (U \otimes V)|\psi\rangle$ is entangled.

Solution 2.4. This is almost by definition. Try to formalize the argument.